which we may integrate to

$$a(t - t_h) = \ln\left(\frac{f}{1-f}\right) \tag{6.85}$$

Exponentiating

$$e^{a(t-t_h)} = \frac{f}{1-f} \tag{6.86}$$

and solving for f, we find

$$f(t) = \frac{e^{a(t-t_h)}}{1+e^{a(t-t_h)}}. \tag{6.87}$$

Example 6.1 (The Callan-Symanzik Equation) Let us consider a quantum field theory on a lattice in which the strength of the non-linear interactions depends upon a single dimensionless coupling constant g. The spacing a of the lattice regulates the infinities, which return as $a \to 0$. The value of an observable P computed on this lattice will depend upon the lattice spacing a and on the coupling constant g, and so will be a function $P(a, g)$ of these two parameters. The "right" value of the coupling constant is the value that makes the result of the computation be as close as possible to the physical value P. Thus, the "right" coupling constant is not a constant at all, but rather a function $g(a)$ that varies with the lattice spacing or cut-off a. Thus as we vary the lattice spacing and go to the continuum limit in which $a \to 0$, we must adjust the coupling function $g(a)$ so that what we compute, $P(a, g(a))$, is equal to the physical value P. That is, $g(a)$ must vary with a so as to keep $P(a, g(a)) = P$. But then $P(a, g(a))$ must remain constant as a varies, so

$$\frac{dP(a, g(a))}{da} = 0. \tag{6.88}$$

Writing this condition as a dimensionless derivative

$$a\,\frac{dP(a, g(a))}{da} = \frac{da}{d\ln a}\,\frac{dP(a, g(a))}{da} = \frac{dP(a, g(a))}{d\ln a} = 0 \tag{6.89}$$

we arrive at the **Callan-Symanzik equation**

$$0 = \frac{dP(a, g(a))}{d\ln a} = \left(\frac{\partial}{\partial \ln a} + \frac{dg}{d\ln a}\,\frac{\partial}{\partial g}\right) P(a, g(a)). \tag{6.90}$$

The coefficient of the second partial derivative (with a minus sign)

$$\beta(g) \equiv -\frac{dg}{d\ln a} \tag{6.91}$$

is called the β-function.

In $SU(N)$ gauge theory, the first two terms of the β-function for small g are

$$\beta(g) = -\beta_0 \, g^3 - \beta_1 \, g^5 \tag{6.92}$$

where

$$\begin{aligned} \beta_0 &= \frac{1}{(4\pi)^2}\left(\frac{11}{3}N - \frac{2}{3}n_f\right) \\ \beta_1 &= \frac{1}{(4\pi)^4}\left(\frac{34}{3}N^2 - \frac{10}{3}Nn_f - \frac{N^2-1}{N}n_f\right) \end{aligned} \tag{6.93}$$

in which n_f is the number of quark flavors. In quantum chromodynamics, $N = 3$.

Combining the definition (6.91) of the β-function with its expansion (6.92) for small g, one arrives at the differential equation

$$\frac{dg}{d\ln a} = \beta_0 \, g^3 + \beta_1 \, g^5 \tag{6.94}$$

which one may integrate

$$\int d\ln a = \ln a + c = \int \frac{dg}{\beta_0 g^3 + \beta_1 g^5} = -\frac{1}{2\beta_0 g^2} + \frac{\beta_1}{2\beta_0^2} \ln\left(\frac{\beta_0 + \beta_1 g^2}{g^2}\right) \tag{6.95}$$

to find

$$a(g) = d\left(\frac{\beta_0 + \beta_1 g^2}{g^2}\right)^{\beta_1/2\beta_0^2} e^{-1/2\beta_0 g^2} \tag{6.96}$$

in which d is a constant of integration. The term $\beta_1 g^2$ is of higher order in g, and if one drops it and absorbs a factor of β_0^2 into a new constant of integration Λ, then one gets

$$a(g) = \frac{1}{\Lambda}\left(\beta_0 g^2\right)^{-\beta_1/2\beta_0^2} e^{-1/2\beta_0 g^2}. \tag{6.97}$$

As $g \to 0$, the lattice spacing $a(g)$ goes to zero *very fast* (as long as $n_f < 17$ for $N = 3$). The inverse of this relation (6.97) is

$$g(a) \approx \left[\beta_0 \ln(a^{-2}\Lambda^{-2}) + (\beta_1/\beta_0)\ln\left(\ln(a^{-2}\Lambda^{-2})\right)\right]^{-1/2}. \tag{6.98}$$

It shows that the coupling constant slowly goes to zero with a, which is a lattice version of **asymptotic freedom**. □