Warning: these notes have been translated from the normal metric to the Peskin metric $(+,-,-,-)$.

Second homework assignment:

(1) Show that if $\phi(x) \equiv \phi_\ell(x)$ is any 4-component field that satisfies the Klein-Gordon equation

$$\left(\Box + m^2\right) \phi_\ell(x) = 0 \tag{0.1}$$

then the field (typo fixed)

$$\psi_\ell(x) \equiv (i\partial\!\!\!/ + m)_{\ell m} \, \phi_m(x) \tag{0.2}$$

satisfies the Dirac equation

$$(i\gamma^a \partial_a - m)\, \psi \equiv (i\partial\!\!\!/ - m)\, \psi = 0. \tag{0.3}$$

(2) The standard boost is the 4×4 matrix

$$L(p) = \exp\left(\alpha \hat{\boldsymbol{p}} \cdot \boldsymbol{B}\right) \tag{0.4}$$

in which the $\boldsymbol{B}$ are the 3 boost matrices defined in Eq.(0.35) of the on-line extract from my book. Show that α must satisfy $\cosh\alpha = p^0/m$ and $\sinh\alpha = |\boldsymbol{p}|/m$. You may consider the case $p = (p^0, 0, 0, |\boldsymbol{p}|)$.

(3) The matrix $D^{(1/2,0)}(L(p))$ that represents the standard boost $L(p)$ in the $(1/2, 0)$ representation of the Lorentz group is the 2×2 matrix

$$D^{(1/2,0)}(L(p)) = \exp\left(-i\alpha \hat{\boldsymbol{p}} \cdot \boldsymbol{K}\right) = \exp\left(-\alpha \hat{\boldsymbol{p}} \cdot \boldsymbol{\sigma}/2\right). \tag{0.5}$$

Show that this matrix is

$$\exp\left(-\alpha \hat{\boldsymbol{p}} \cdot \boldsymbol{\sigma}/2\right) = \cosh(\alpha/2) - \hat{\boldsymbol{p}} \cdot \boldsymbol{\sigma} \, \sinh(\alpha/2). \tag{0.6}$$

(4) Show further that

$$\cosh\left(\frac{\alpha}{2}\right) = \sqrt{\frac{p^0 + m}{2m}} \tag{0.7}$$

and that

$$\sinh\left(\frac{\alpha}{2}\right) = \sqrt{\frac{p^0 - m}{2m}} \tag{0.8}$$

so that (typo fixed)

$$\exp\left(-\alpha \hat{\boldsymbol{p}} \cdot \boldsymbol{\sigma}/2\right) = \frac{p^0 + m - \boldsymbol{p} \cdot \boldsymbol{\sigma}}{\sqrt{2m(p^0 + m)}}. \tag{0.9}$$

(5) The matrix $D^{(1/2,0)}(L(p))$ that represents the standard boost $L(p)$ in the $(0,1/2)$ representation of the Lorentz group is the 2×2 matrix

$$D^{(0,1/2)}(L(p)) = \exp\left(i\alpha\hat{\boldsymbol{p}} \cdot \boldsymbol{K}\right) = \exp\left(\alpha\hat{\boldsymbol{p}} \cdot \boldsymbol{\sigma}/2\right). \tag{0.10}$$

Show that

$$\exp\left(\alpha\hat{\boldsymbol{p}} \cdot \boldsymbol{\sigma}/2\right) = \frac{p^0 + m + \hat{\boldsymbol{p}} \cdot \boldsymbol{\sigma}}{\sqrt{2m(p^0+m)}} \tag{0.11}$$

and that therefore the boost in the Dirac representation is

$$D^D(L(p)) = \frac{m + \not{p}\,\gamma^0}{\sqrt{2m(p^0+m)}} \equiv \frac{m + p_a\gamma^a\,\gamma^0}{\sqrt{2m(p^0+m)}}. \tag{0.12}$$