Solutions to 2d HW

(1) We assume ϕ_ℓ $\ell = 1-4$ satisfies

$$(\Box + m^2)\phi_\ell(x) = 0.$$

Let $\quad \psi_\ell(x) = (i\not\partial + m)_{\ell m}\,\phi_m(x).$

Then ψ satisfies the Dirac equation

$$
\begin{aligned}
(i\not\partial - m)\psi &= (i\not\partial - m)(i\not\partial + m)\phi \\
&= \left(-\gamma^a\gamma^b\partial_a\partial_b - m^2\right)\phi \\
&= -\left(\tfrac{1}{2}\{\gamma^a,\gamma^b\}\partial_a\partial_b + m^2\right)\phi \\
&= -\left(\eta^{ab}\partial_a\partial_b + m^2\right)\phi = -(\Box + m^2)\phi = 0.
\end{aligned}
$$

(2) Take $\hat{p} = \hat{z}$. Then

$$L(p) = e^{\alpha B_3} = 1 + \alpha B_3 + \frac{\alpha^2}{2}B_3^2 + \dots \frac{\alpha^n}{n!}B_3^n + \dots$$

$$B_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & & & \\ 0 & & & \\ 1 & & & \end{pmatrix}, \qquad B_3^2 = \begin{pmatrix} 1 & & & \\ & & & \\ & & & \\ & & & 1 \end{pmatrix}, \quad \text{so}$$

$$L(p_z) = \begin{pmatrix} \sum \frac{\alpha^{2n}}{(2n)!} & 0 & 0 & \sum_{n=0}^{\infty}\frac{\alpha^{2n+1}}{(2n+1)!} \\ 0 & & & \\ 0 & & & \\ \sum_{n=0}^{\infty}\frac{\alpha^{2n+1}}{(2n+1)!} & & & \sum_{n=0}^{\infty}\frac{\alpha^{2n}}{(2n)!} \end{pmatrix} = \begin{pmatrix} \cosh\alpha & 0 & 0 & \sinh\alpha \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sinh\alpha & 0 & 0 & \cosh\alpha \end{pmatrix}.$$

We need

$$\begin{pmatrix} p^0 \\ 0 \\ 0 \\ |\vec{p}| \end{pmatrix} = L(p)\begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} m\cosh\alpha \\ 0 \\ 0 \\ m\sinh\alpha \end{pmatrix} \quad \text{so} \quad \begin{array}{l} p^0 = m\cosh\alpha \\ |\vec{p}| = m\sinh\alpha . \end{array}$$

(3) $$D^{(1/2,0)}(L(p)) = e^{-\alpha\hat{p}\cdot\sigma/2} = 1 - \alpha\frac{\hat{p}\cdot\sigma}{2} + \frac{(\alpha/2)^2(\hat{p}\cdot\sigma)^2}{2!} + \cdots$$

$$= 1 - \frac{\alpha}{2}\hat{p}\cdot\sigma + \frac{(\alpha/2)^2}{2} + \frac{(-\alpha/2)^3}{3!}\hat{p}\cdot\sigma + \cdots$$

$$= \sum_{n=0}^{\infty}\frac{(\alpha/2)^{2n}}{(2n)!} - \sum_{n=0}^{\infty}\frac{(\alpha/2)^{2n+1}}{(2n+1)!}\frac{\hat{p}\cdot\sigma}{2}$$

$$= \cosh\alpha/2 - \hat{p}\cdot\sigma\sinh\alpha/2 .$$

(4) We use the half-angle formulas

$$\cosh\alpha/2 = \sqrt{\frac{\cosh\alpha + 1}{2}} = \sqrt{\frac{p^0/m + 1}{2}} = \sqrt{\frac{p^0+m}{2m}}$$

and (for $\alpha > 0$)

$$\sinh\alpha/2 = \sqrt{\frac{\cosh\alpha - 1}{2}} = \sqrt{\frac{p^0/m - 1}{2}} = \sqrt{\frac{p^0 - m}{2m}} .$$

So $$D^{(1/2,0)}(L(p)) = e^{-\alpha\hat{p}\cdot\vec{\sigma}/2} = \sqrt{\frac{p^0+m}{2m}} - \hat{p}\cdot\vec{\sigma}\sqrt{\frac{p^0-m}{2m}} .$$

Now

$$\frac{p^0-m}{2m} = \frac{(p^0-m)(p^0+m)}{2m(p^0+m)} = \frac{p^{0\,2}-m^2}{2m(p^0+m)} = \frac{|\vec{p}|^2}{2m(p^0+m)}$$

and

$$\frac{p^0+m}{2m} = \frac{(p^0+m)^2}{2m(p^0+m)}, \quad \text{so}$$

$$D^{(1/2,0)}(L(p)) = \sqrt{\frac{(p^0+m)^2}{2m(p^0+m)}} - \hat{p}\cdot\vec{\sigma}\sqrt{\frac{|\vec{p}|^2}{2m(p^0+m)}}$$

$$= \frac{p^0+m-\vec{p}\cdot\sigma}{\sqrt{2m(p^0+m)}}.$$

(5) Well,

$$D^{(0,1/2)}(L(p)) = e^{\alpha\hat{p}\cdot\sigma/2} = \cosh\alpha/2 + \hat{p}\cdot\sigma\sinh\alpha/2$$

$$= \frac{p^0+m+\vec{p}\cdot\vec{\sigma}}{\sqrt{2m(p^0+m)}}, \quad \text{so}$$

$$D^{D}(L(p)) = D^{(1/2,0)}(L(p)) \oplus D^{(0,1/2)}(L(p)) = \begin{pmatrix} D^{(1/2,0)}(L(p)) & 0 \\ 0 & D^{(0,1/2)}(L(p)) \end{pmatrix}$$

So

$$D^{D}(L(p)) = \begin{pmatrix} p^0 + m - \vec{p}\cdot\vec{\sigma} & 0 \\ 0 & p^0 + m + \vec{p}\cdot\vec{\sigma} \end{pmatrix} \frac{1}{\sqrt{2m(p^0+m)}}.$$

Now $\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ and $\vec{\gamma} = \begin{pmatrix} 0 & +\vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$, so

$$\begin{pmatrix} p^0 & 0 \\ 0 & p^0 \end{pmatrix} = \gamma^{0\,2} p^0 \quad \text{and} \quad \begin{pmatrix} -\vec{p}\cdot\sigma & 0 \\ 0 & \vec{p}\cdot\sigma \end{pmatrix} = \begin{pmatrix} 0 & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & 0 \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

so

$$\begin{pmatrix} p^0 - \vec{p}\cdot\sigma & 0 \\ 0 & p^0 + \vec{p}\cdot\sigma \end{pmatrix} = \left(p^0\gamma^0 - \vec{p}\cdot\vec{\gamma} \right)\gamma^0$$

$$= p_a\gamma^a\gamma^0 = \not{p}\gamma^0. \qquad \text{so}$$

$$D^{D}(L(p)) = \frac{(m + \not{p}\gamma^0)}{\sqrt{2m(p^0+m)}}.$$