VII.5 Grand Unification

Crying out for unification

A gauge theory is specified by a group and the representations the matter fields belong to. Let us go back to chapter VII.2 and make a catalogue for the $SU(3) \otimes SU(2) \otimes U(1)$ theory. For example, the left handed up and down quarks are in a doublet $\begin{pmatrix} u^\alpha \\ d^\alpha \end{pmatrix}_L$ with hypercharge $\frac{1}{2}Y = \frac{1}{6}$. Let us denote this by $(3, 2, \frac{1}{6})_L$, with the three numbers indicating how these fields transform under $SU(3) \otimes SU(2) \otimes U(1)$. Similarly, the right handed up quark is $(3, 1, \frac{2}{3})_R$. The leptons are $(1, 2, -\frac{1}{2})_L$ and $(1, 1, -1)_R$, where the "1" in the first entry indicates that these fields do not participate in the strong interaction. Writing it all down, we see that the quarks and leptons of each family are placed in

$$(3, 2, \tfrac{1}{6})_L,\ (3, 1, \tfrac{2}{3})_R,\ (3, 1, -\tfrac{1}{3})_R,\ (1, 2, -\tfrac{1}{2})_L, \text{ and } (1, 1, -1)_R \tag{1}$$

This motley collection of representations practically cries out for further unification. Who would have constructed the universe by throwing this strange looking list down?

What we would like to have is a larger gauge group G containing $SU(3) \otimes SU(2) \otimes U(1)$, such that this laundry list of representations is unified into (ideally) one great big representation. The gauge bosons in G [but not in $SU(3) \otimes SU(2) \otimes U(1)$ of course] would couple the representations in (1) to each other.

Before we start searching for G, note that since gauge transformations commute with the Lorentz group, these desired gauge transformations cannot change left handed fields to right handed fields. So let us change all the fields in (1) to left handed fields. Recall from exercise II.1.9 that charge conjugation changes left handed fields to right handed fields and vice versa. Thus, instead of (1) we can write

$$(3, 2, \tfrac{1}{6}),\ (3^*, 1, -\tfrac{2}{3}),\ (3^*, 1, \tfrac{1}{3}),\ (1, 2, -\tfrac{1}{2}), \text{ and } (1, 1, 1) \tag{2}$$

We now omit the subscripts L and R: everybody is left handed.

A perfect fit

The smallest group that contains $SU(3) \otimes SU(2) \otimes U(1)$ is $SU(5)$. (If you are shaky about group theory, study appendix B now.) Recall that $SU(5)$ has $5^2 - 1 = 24$ generators. Explicitly, the generators are represented by 5 by 5 hermitean traceless matrices acting on five objects we denote by ψ^μ with $\mu = 1, 2, \ldots, 5$. [These five objects form the fundamental or defining representation of $SU(5)$.]

It is now obvious how we can fit $SU(3)$ and $SU(2)$ into $SU(5)$. Of the 24 matrices that generate $SU(5)$, eight have the form $\left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and three the form $\left(\begin{smallmatrix} 0 & 0 \\ 0 & B \end{smallmatrix}\right)$, where A represents 3 by 3 hermitean traceless matrices (of which there are $3^2 - 1 = 8$, the so-called Gell-Mann matrices) and B represents 2 by 2 hermitean traceless matrices (of which there are $2^2 - 1 = 3$, namely the Pauli matrices). Clearly, the former generate an $SU(3)$ and the latter an $SU(2)$. Furthermore, the 5 by 5 hermitean traceless matrix

$$\frac{1}{2}Y = \begin{pmatrix} -\frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \tag{3}$$

generates a $U(1)$. Without being coy about it, we have already called this matrix the hypercharge $\frac{1}{2}Y$.

In other words, if we separate the index $\mu = \{\alpha, i\}$ with $\alpha = 1, 2, 3$ and $i = 4, 5$, then the $SU(3)$ acts on the index α and the $SU(2)$ acts on the index i. Thus, the three objects ψ^α transform as a 3-dimensional representation under $SU(3)$ and hence could be a 3 or a 3*. Let us choose ψ^α as transforming as 3; we will see shortly that this is the right choice with $Y/2$ given as in (3). The three objects ψ^α do not transform under $SU(2)$ and hence each of them belongs to the singlet 1 representation. Furthermore, they carry hypercharge $-\frac{1}{3}$ as we can read off from (3). To sum up, ψ^α transform as $(3, 1, -\frac{1}{3})$ under $SU(3) \otimes SU(2) \otimes U(1)$. On the other hand, the two objects ψ^i transform as 1 under $SU(3)$ and 2 under $SU(2)$, and carry hypercharge $\frac{1}{2}$; thus they transform as $(1, 2, \frac{1}{2})$. In other words, we embed $SU(3) \otimes SU(2) \otimes U(1)$ into $SU(5)$ by specifying how the defining representation of $SU(5)$ decomposes into representations of $SU(3) \otimes SU(2) \otimes U(1)$

$$5 \to (3, 1, -\tfrac{1}{3}) \oplus (1, 2, \tfrac{1}{2}) \tag{4}$$

Taking the conjugate we see that

$$5^* \to (3^*, 1, \tfrac{1}{3}) \oplus (1, 2, -\tfrac{1}{2}) \tag{5}$$

Inspecting (2), we see that $(3^*, 1, \frac{1}{3})$ and $(1, 2, -\frac{1}{2})$ appear on the list. We are on the right track! The fields in these two representations fit snugly into 5*.

This accounts for five of the fields contained in (2); we still have the ten fields

$$(3, 2, \tfrac{1}{6}),\ (3^*, 1, -\tfrac{2}{3}), \text{ and } (1, 1, 1) \tag{6}$$

Consider the next representation of $SU(5)$ in order of size, namely the antisymmetric tensor representation $\psi^{\mu\nu}$. Its dimension is $(5 \times 4)/2 = 10$, precisely the number we want, if only the quantum numbers under $SU(3) \otimes SU(2) \otimes U(1)$ work out!

Since we know that $5 \to (3, 1, -\frac{1}{3}) \oplus (1, 2, \frac{1}{2})$, we simply (again, see appendix B!) have to work out the antisymmetric product of $(3, 1, -\frac{1}{3}) \oplus (1, 2, \frac{1}{2})$ with itself, namely the direct sum of (where $\otimes_A$ denotes the antisymmetric product)

$$(3, 1, -\tfrac{1}{3}) \otimes_A (3, 1, -\tfrac{1}{3}) = (3^*, 1, -\tfrac{2}{3}) \tag{7}$$

$$(3, 1, -\tfrac{1}{3}) \otimes_A (1, 2, \tfrac{1}{2}) = (3, 2, -\tfrac{1}{3} + \tfrac{1}{2}) = (3, 2, \tfrac{1}{6}) \tag{8}$$

and

$$(1, 2, \tfrac{1}{2}) \otimes_A (1, 2, \tfrac{1}{2}) = (1, 1, 1) \tag{9}$$

[I will walk you through (7): In $SU(3)$ $3 \otimes_A 3 = 3^*$ (remember ε_{ijk} from appendix B?), in $SU(2)$ $1 \otimes_A 1 = 1$, and in $U(1)$ the hypercharges simply add $-\frac{1}{3} - \frac{1}{3} = -\frac{2}{3}$.]

Lo and behold, these $SU(3) \otimes SU(2) \otimes U(1)$ representations form exactly the collection of representations in (6). In other words,

$$10 \to (3, 2, \tfrac{1}{6}) \oplus (3^*, 1, -\tfrac{2}{3}) \oplus (1, 1, 1) \tag{10}$$

The known quark and lepton fields in a given family fit perfectly into the 5^* and 10 representations of $SU(5)$!

I have just described the $SU(5)$ grand unified theory of Georgi and Glashow. In spite of the fact that the theory has not been directly verified by experiment, it is extremely difficult for me and for many other physicists not to believe that $SU(5)$ is at least structurally correct, in view of the perfect group theoretic fit.

It is often convenient to display the contents of the representation 5^* and 10, using the names given to the various fields historically. We write 5^* as a column vector

$$\psi_\mu = \begin{pmatrix} \psi_\alpha \\ \psi_i \end{pmatrix} = \begin{pmatrix} \bar{d}_\alpha \\ \nu \\ e \end{pmatrix} \tag{11}$$

and the 10 as an antisymmetric matrix

$$\psi^{\mu\nu} = \{\psi^{\alpha\beta}, \psi^{\alpha i}, \psi^{ij}\}$$
$$= \begin{pmatrix} 0 & \bar{u} & -\bar{u} & d & u \\ -\bar{u} & 0 & \bar{u} & d & u \\ \bar{u} & -\bar{u} & 0 & d & u \\ -d & -d & -d & 0 & \bar{e} \\ -u & -u & -u & -\bar{e} & 0 \end{pmatrix} \tag{12}$$

(I suppressed the color indices.)

Deepening our understanding of physics

Aside from its esthetic appeal, grand unification deepens our understanding of physics enormously.

1. Ever wondered why electric charge is quantized? Why don't we see particles with charge equal to $\sqrt{\pi}$ times the electron's charge? In quantum electrodynamics, you could perfectly well write down

$$\mathcal{L} = \bar{\psi}[i(\partial\!\!\!/ - iA\!\!\!/) - m]\psi + \bar{\psi}'[i(\partial\!\!\!/ - i\sqrt{\pi}\,A\!\!\!/) - m']\psi' + \cdots \tag{13}$$

In contrast, in grand unified theory A_μ couples to a generator of the grand unifying gauge group, and you know that the generators of any group such as $SU(N)$ (that is not given by the direct product of $U(1)$ with other groups) are forced by the nontrivial commutation relations $[T_a, T_b] = i f_{abc} T_c$ to assume quantized values. For example, the eigenvalues of T_3 in $SU(2)$, which depend on the representation of course, must be multiples of $\frac{1}{2}$. Within $SU(3) \times SU(2) \times U(1)$, we cannot understand charge quantization: The generator of $U(1)$ is not quantized. But upon grand unification into $SU(5)$ [or more generally any group without $U(1)$ factors] electric charge is quantized.

The result here is deeply connected to Dirac's remark (chapter IV.4) that electric charge is quantized if the magnetic monopole exists. We know from chapter V.7 that spontaneously broken nonabelian gauge theories such as the $SU(5)$ theory contain the monopole.

2. Ever wondered why the proton charge is exactly equal and opposite to the electron charge? This important fact allows us to construct the universe as we know it. Atoms must be electrically neutral to some fantastic degree of accuracy for standard cosmology to work; otherwise, electrostatic forces between macroscopic matter would tear the universe apart.

This remarkable fact is nicely incorporated into $SU(5)$. It is fun to see how it goes. Evaluating tr $Q = 0$ over the 5^* implies that $3Q_{\bar{d}} = -Q_{e^-}$. I have used the fact that the strong interaction commutes with electromagnetism and hence quarks with different color have the same charge. Now let us calculate the proton charge Q_P:

$$Q_P = 2Q_u + Q_d = 2(Q_d + 1) + Q_d = 3Q_d + 2 = Q_{e^-} + 2 \tag{14}$$

If $Q_{e^-} = -1$, then $Q_P = -Q_{e^-}$, as is indeed the case!

3. Recall that in electroweak theory we defined $\tan\theta = g_1/g_2$, with the coupling of the gauge bosons $g_2 A^a_\mu T_a + g_1 B_\mu (Y/2)$. Since the normalization of A^a_μ and B_μ is fixed by their respective kinetic energy term, the relative strength of g_2 and g_1 is determined by the normalization of $Y/2$ relative to T_3. Let us evaluate tr T_3^2 and $\text{tr}(Y/2)^2$ on the defining representation 5 : tr $T_3^2 = (\frac{1}{2})^2 + (\frac{1}{2})^2 = \frac{1}{2}$ and $\text{tr}(Y/2)^2 = (\frac{1}{3})^2 3 + (\frac{1}{2})^2 2 = \frac{5}{6}$.

Thus, T_3 and $\sqrt{3/5}(Y/2)$ are normalized equally. So the correct grand unified combination is $A^a_\mu T_a + B_\mu\sqrt{3/5}(Y/2)$, and therefore $\tan\theta = g_1/g_2 = \sqrt{3/5}$ or

$$\sin^2\theta = \frac{3}{8} \tag{15}$$

at the grand unification scale. To compare with the experimental value of $\sin^2\theta$ we would have to study how the couplings g_2 and g_1 flow under the renormalization group down to low energies. We will postpone this discussion until the next chapter.

Freedom from anomaly

Recall from chapter VII.2 that the key to proving renormalizability of nonabelian gauge theory is the ability to pass freely between the unitary gauge and the R_ξ gauge. The crucial ingredient is gauge invariance and the resulting Ward-Takahashi identities (see chapter II.7).

Suddenly you start to worry. What about the chiral anomaly? The existence of the anomaly means that some Ward-Takahashi identities fail to hold. For our theories to make sense, they had better be free from anomalies. I remarked in chapter IV.7 that the historical name "anomaly" makes it sound like some kind of sickness. Well, in a way, it is.

We should have already checked the $SU(3) \otimes SU(2) \otimes U(1)$ theory for anomalies, but we didn't. I will let you do it as an exercise. Here I will show that the $SU(5)$ theory is healthy. If the $SU(5)$ theory is anomaly-free, then a fortiori so is the $SU(3) \otimes SU(2) \otimes U(1)$ theory.

In chapter IV.7 I computed the anomaly in an abelian theory but as I remarked there clearly all we have to do to generalize to a nonabelian theory is to insert a generator T_a of the gauge group at each vertex of the triangle diagram in figure IV.7.1. Summing over the various fermions running around the loop, we see that the anomaly is proportional to $A_{abc}(R) \equiv \mathrm{tr}(T_a\{T_b, T_c\})$, where R denotes the representation to which the fermions belong. We have to sum $A_{abc}(R)$ over all the representations in the theory, remembering to associate opposite signs to left handed and right handed fermion fields. (It may be helpful to remind yourself of remark 3 in chapter IV.7 and exercise IV.7.6.)

We are now ready to give the $SU(5)$ theory a health check. First, all fermion fields in (2) are left handed. Second, convince yourself (simply imagine calculating A_{abc} for all possible abc) that it suffices to set T_a, T_b, and T_c all equal to

$$T \equiv \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix}$$

a multiple of the hypercharge. Let us now evaluate $\mathrm{tr}\, T^3$ on the 5^* representation,

$$\mathrm{tr}\, T^3|_{5^*} = 3(-2)^3 + 2(+3)^3 = 30 \tag{16}$$

and on the 10,

$$\mathrm{tr}\, T^3|_{10} = 3(+4)^3 + 6(-1)^3 + (-6)^3 = -30 \tag{17}$$

An apparent miracle! The anomaly cancels.

This remarkable cancellation between sums of cubes of a strange list of numbers suggests strongly, to say the least, that $SU(5)$ is not the end of the story. Besides, it would be nice if the 5^* and 10 could be unified into a single representation.

Exercises

VII.5.1 Write down the charge operator Q acting on 5, the defining representation ψ^{μ}. Work out the charge content of the $10 = \psi^{\mu\nu}$ and identify the various fields contained therein.

VII.5.2 Show that for any grand unified theory, as long as it is based on a simple group, we have at the unification scale

$$\sin^2\theta = \frac{\sum T_3^2}{\sum Q^2} \tag{18}$$

where the sum is taken over all fermions.

VII.5.3 Check that the $SU(3)\otimes SU(2)\otimes U(1)$ theory is anomaly-free. [Hint: The calculation is more involved than in $SU(5)$ since there are more independent generators. First show that you only have to evaluate $\text{tr}\, Y\{T_a, T_b\}$ and $\text{tr}\, Y^3$, with T_a and Y the generators of $SU(2)$ and $U(1)$, respectively.]

VII.5.4 Construct grand unified theories based on $SU(6)$, $SU(7)$, $SU(8)$, ..., until you get tired of the game. People used to get tenure doing this. [Hint: You would have to invent fermions yet to be experimentally discovered.]