Form Factors

$$\langle p', s' | J^\mu(x) | p, s\rangle = \langle p', s' | e^{-iP\cdot x} J^\mu(0) e^{iP\cdot x} | p, s\rangle$$

$$= e^{i(p-p')\cdot x} \langle p' s' | J^\mu(0) | p, s\rangle .$$

Since $\partial_\mu J^\mu(x) = 0$, we have

$$0 = e^{i(p-p')x} (p-p')_\mu \langle p' s' | J^\mu(0) | p, s\rangle \quad \text{so}$$

$$0 = (p-p')_\mu \langle p', s' | J^\mu(0) | p, s\rangle .$$

Also, for $\mu = 0$

$$\langle p' s' | Q | p s\rangle = \int \langle p' s' | J^0(x) | p s\rangle \, d^3x$$

$$= \int e^{i(p-p')\cdot x} \langle p' s' | J^0(0) | p s\rangle \, d^3x$$

$$= (2\pi)^3 \delta^3(p-p') \langle p' s' | J^0(0) | p s\rangle$$

$$= q \langle p' s' | p s\rangle = q\, \delta^3(p-p')\, \delta_{ss'}$$

so $$\langle p s' | J^0(0) | p s\rangle = \frac{q\, \delta_{ss'}}{(2\pi)^3} .$$

spin-zero case

$$\langle p'|J^\mu(0)|p\rangle = \frac{q\,\mathcal{J}^\mu(p';p)}{(2\pi)^3\sqrt{2p'^0\,2p^0}}$$

Now p' and p are the 4-momenta of physical particles, so $p'^2 = p^2 = -m^2$. So

$(p \pm p')^2 = -2m^2 \pm 2\,p \cdot p'$ and so with $k = p - p'$

$$\mathcal{J}^\mu(p',p) = (p'+p)^\mu F(k^2) + i(p'-p)^\mu H(k^2).$$

But $0 = (p'-p)_\mu \mathcal{J}^\mu = (p'-p)\cdot(p'+p)F$

$+ i\,(p'-p)^2 H$

$= i\,k^2 H(k^2) = 0.$ So $H(k^2) = 0$.

So $\mathcal{J}^\mu(p',p) = (p'+p)^\mu F(k^2)$

$$\langle p|J^0(0)|p\rangle = \frac{q}{(2\pi)^3} = \frac{q\,2p^0 F(0)}{(2\pi)^3\,2p^0} = \frac{q\,F(0)}{(2\pi)^3}$$

So $F(0) = 1$. This $F(k^2)$ is the electromagnetic form factor of the scalar particle in question.

Spin $\frac{1}{2}$

Lorentz invariance now gives

$$\langle p's'|J^\mu(0)|p,s\rangle = iq\,\frac{\bar{u}(p',s')}{(2\pi)^3}\,\Gamma^\mu(p',p)\,u(p,s)$$

where Γ^μ is some combination of 1, γ^μ, $[\gamma^\mu,\gamma^\nu]$, $\gamma_5\gamma^\mu$, and γ_5. But because

$$(i\not{p}+m)\,u(p,s) = 0 = \bar{u}(p',s')\,(i\not{p}'+m)$$

we can reduce $\bar{u}\Gamma^\mu u$ to

$$\bar{u}(p',s')\,\Gamma^\mu(p',p)\,u(p,s) = \bar{u}(p',s')\Big[\gamma^\mu F(k^2)$$

$$-\frac{i}{2m}(p+p')^\mu G(k^2) + \frac{(p-p')^\mu}{2m}H(k^2)\Big]u(p,s).$$

For instance $p^\mu \not{p} u = im\,p^\mu u(p,s)$ and

$$\bar{u}(p',s')\,p'^\mu \not{p}' = p'^\mu\, im\,\bar{u}(p',s').$$

Now $J^{\mu\dagger}(0) = J^\mu(0)$, so

$$\langle p',s'|J^\mu(0)|p,s\rangle^* = \langle p\,s|\,J^\mu(0)\,|p',s'\rangle \text{ or}$$

$$-u^\dagger\,\Gamma^{\mu\dagger}_{(p',p)}\,\beta^\dagger u' = \bar{u}\,\Gamma^\mu_{(p,p')}u' = -u^\dagger\,\Gamma^{\mu\dagger}_{(p',p)}\,\beta\,u'$$

since $\gamma^{0\dagger} = -\gamma^0$ and $\beta = i\gamma^0$ so $\beta^\dagger = \beta$ and $\beta^2 = 1$.

So $\quad -\Gamma^{\mu\dagger}(p',p)\beta = \beta\Gamma^{\mu}(p,p')$ $\quad$ so

$$\beta\Gamma^{\mu\dagger}(p',p)\beta = -\Gamma^{\mu}(p,p')$$

That means

$$\beta\left[\gamma^{\mu\dagger}F^* + \frac{i}{2m}(p+p')^\mu G^* + \frac{(p-p')^\mu}{2m}H^*\right]\beta$$

$$= -\left[\gamma^\mu F - \frac{i}{2m}(p+p')^\mu G - \frac{(p-p')^\mu}{2m}H\right].$$

$\gamma^{0\dagger} = -\gamma^0$ $\quad$ so $\quad \beta\gamma^{0\dagger}\beta = -\beta\gamma^0\beta = -\gamma^0$

$\vec{\gamma}^\dagger = \vec{\gamma}$ $\quad$ so $\quad \beta\vec{\gamma}^\dagger\beta = \beta\vec{\gamma}\beta = -\vec{\gamma}$

So $\quad F^* = F, \quad G^* = G$, and $\quad H^* = H$.

Current conservation implies

$$0 = (p-p')_\mu\,\bar{u}'\,\Gamma^\mu(p',p)u$$

$$= \bar{u}'\left[(\not{p}-\not{p}')F - \frac{i}{2m}(p^2-p'^2)G + \frac{k^2}{2m}H\right]u$$

$$= \bar{u}'\left[(im-im)F + 0\cdot G + \frac{k^2}{2m}H\right]u = \frac{k^2}{2m}\,\bar{u}'Hu.$$

So $\qquad H(k^2) = 0.$

when $p' \to p$,

$$\langle p's' | J^\mu(0) | ps \rangle = \frac{iq}{(2\pi)^3} \bar{u}(p,s') \Gamma^\mu(p,p) u(p,s)$$

$$= \frac{iq}{(2\pi)^3} \bar{u}(p,s') \left[\gamma^\mu F(0) - \frac{i}{m} p^\mu G(0) \right] u(p,s)$$

But $\{ \gamma^\mu, i\gamma^\nu p_\nu + m \} = 2i\eta^{\mu\nu} p_\nu + 2m\gamma^\mu$

$$= 2m\gamma^\mu + 2ip^\mu,$$

so since $(i\not{p} + m)u = 0 = \bar{u}(i\not{p} + m)$,

$$\bar{u}(p,s')\gamma^\mu u(p,s) = -i \frac{p^\mu}{m} \bar{u}(p,s')u(p,s).$$

Thus since $\bar{u}(p,s')u(p,s) = \delta_{ss'} m/p^0$

$$\langle p,s' | J^\mu(0) | p,s \rangle = \frac{iq}{(2\pi)^3} \bar{u}(p,s') \left[-\frac{i p^\mu}{m} F(0) - \frac{i}{m} p^\mu G(0) \right] u(p,s)$$

$$= \frac{q}{(2\pi)^3} \frac{p^\mu}{m} \frac{m}{p^0} \delta_{ss'} \left[F(0) + G(0) \right]$$

Recall
$\langle \vec{p}, s' | J^0(0) | \vec{p}, s \rangle = q\, \delta_{ss'} / (2\pi)^3$
so

$$= \frac{q}{(2\pi)^3} \frac{p^\mu}{p^0} \delta_{ss'} \left[F(0) + G(0) \right] \Rightarrow \frac{q}{(2\pi)^3} \delta_{s's} \text{ for } \mu = 0$$

so $$F(0) + G(0) = 1.$$

Since $i\not{p}\, u(p,s) = -m\, u(p,s)$ and $\bar{u}(p',s')\, i\not{p}' = -im\, \bar{u}(p',s')$,
one has

$$\bar{u}(p',s')\frac{i}{2}[\gamma^\mu,\gamma^\nu](p'-p)_\nu u(p,s)$$

$$= \bar{u}'\frac{i}{2}[\gamma^\mu, \not{p}'-\not{p}]u = \bar{u}'\Big(-i\not{p}'\gamma^\mu + \frac{i}{2}\{\gamma^\mu,\not{p}'\}$$
$$-i\gamma^\mu\not{p} + \frac{i}{2}\{\gamma^\mu,\not{p}\}\Big)u$$

$$= \bar{u}'\left(m\gamma^\mu + ip'^\mu + m\gamma^\mu + ip^\mu\right)u$$

$$= \bar{u}(p',s')\left[i(p'+p)^\mu + 2m\gamma^\mu\right]u(p,s).$$

Incidentally, this relation explains why in the parametrization

$$\bar{u}'\Gamma^\mu(p',p)u = \bar{u}'(p',s')\Big[\gamma^\mu F_1(k^2) + \frac{i}{2}[\gamma^\mu,\gamma^\nu](p'-p)_\nu F_2(k^2)\Big]u(p,s)$$

works with

$$F(k^2) = F_1(k^2) + 2m F_2(k^2)$$
$$G(k^2) = -2m\, F_2(k^2) \text{ and with } F_1(0) = 1.$$

We'll use it to write

$$\bar{u}'\left[\gamma^\mu F - \frac{i}{2m}(p+p')^\mu G\right]u = \bar{u}'\Big[-\frac{i}{2m}(p+p')^\mu(F+G)$$
$$+\frac{i}{4m}[\gamma^\mu,\gamma^\nu](p'-p)_\nu F\Big]u.$$

$$\mathcal{J}^{ij} = -\frac{i}{4}[\gamma^i, \gamma^j] = \frac{1}{2}\epsilon_{ijk}\begin{pmatrix}\sigma_k & 0 \\ 0 & \sigma_k\end{pmatrix}$$

and

$$\mathcal{J}^{i0} = -\frac{i}{4}[\gamma^i, \gamma^0] = \frac{i}{2}\begin{pmatrix}\sigma_i & 0 \\ 0 & -\sigma_i\end{pmatrix}$$

where $\gamma^0 = -i\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}$ and $\gamma^i = -i\begin{pmatrix}0 & \sigma_i \\ -\sigma_i & 0\end{pmatrix}$

Now we go to the zero-momentum limit $\vec{p} = \vec{p}\,' = \vec{0}$ in which

$$u(\vec{0}, +) = \frac{1}{\sqrt{2}}\begin{pmatrix}1 \\ 0 \\ 1 \\ 0\end{pmatrix} \qquad u(\vec{0}, -) = \frac{1}{\sqrt{2}}\begin{pmatrix}0 \\ 1 \\ 0 \\ 1\end{pmatrix}$$

$\bar{u} = u^\dagger \beta = u^\dagger i\gamma^0 = u^\dagger\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix} = u^\dagger$ for $\vec{p} = \vec{p}\,' = \vec{0}$.

So $\bar{u}[\gamma^i, \gamma^j]u = 2i\,\epsilon_{ijk}\, u^\dagger\begin{pmatrix}\sigma_k & 0 \\ 0 & \sigma_k\end{pmatrix}u = 4i\,\epsilon_{ijk}\, u^\dagger\begin{pmatrix}\frac{\sigma_k}{2} & 0 \\ 0 & \frac{\sigma_k}{2}\end{pmatrix}u$

$$= 4i\,\epsilon_{ijk}\left(\frac{\sigma_k}{2}\right)_{s's}.$$

And $\bar{u}[\gamma^i, \gamma^0]u = -2\, u^\dagger\begin{pmatrix}\sigma_i & 0 \\ 0 & -\sigma_i\end{pmatrix}u = 0.$

So

$$\bar{u}(p', s')\Gamma^k(p', p)u(p, s) \approx -\frac{i}{2m}u^\dagger u\,(p + p')^k\underbrace{(F(0) + G(0))}_{=1}$$

$$+ \frac{i u^\dagger}{4m}[\gamma^k, \gamma^j](p' - p)_j\, F(0)\, u$$

$$= -\frac{i}{2m}(p + p')^k\delta_{s's} - \frac{1}{m}\epsilon_{kjk'}(p' - p)_j\frac{\sigma_{k'\,s's}}{2}F(0)$$

$$\bar{u}(p',s')\vec{\Gamma}\, u(p,s) \simeq \frac{-i}{2m}(\overrightarrow{p+p'})\,\delta_{ss'} + \frac{1}{m}(\overrightarrow{p-p'})\times\left(\frac{\sigma}{2}\right)_{s's} F(0).$$

In a weak magnetic field,

$$H' = -\int \vec{J}(x)\cdot\vec{A}(x)\, d^3x$$

and so

$$\langle p's'|H'|ps\rangle = -\int \langle p's'|\vec{J}(x)|p,s\rangle\cdot\vec{A}(x)\, d^3x$$

$$= -\frac{iq}{(2\pi)^3}\int \bar{u}(p',s')\vec{\Gamma}\, u(p,s)\cdot\vec{A}(x)\, d^3x\, e^{i(p-p')\cdot x}$$

See p. 9 for why we dropped $(\overrightarrow{p+p'})/2m$.

$$= -\frac{iqF(0)}{m(2\pi)^3}\int d^3x\, e^{i(\overrightarrow{p-p'})\cdot\vec{x}}\,\vec{A}(x)\cdot(p-p')\times\left(\frac{\sigma}{2}\right)_{s's}$$

$$= \frac{-iqF(0)}{m(2\pi)^3}\int d^3x\, \vec{A}(x)\cdot\left(-i\vec{\nabla}e^{i(p-p')\cdot x}\right)\times\left(\frac{\sigma}{2}\right)_{s's}$$

$$= \frac{-qF(0)}{m(2\pi)^3}\int d^3x\, A_i(x)\epsilon_{ijk}\partial_j e^{i(p-p')\cdot x}\left(\frac{\sigma_k}{2}\right)_{s's}$$

$$= \frac{-qF(0)}{m(2\pi)^3}\int d^3x\, e^{i(p-p')\cdot x}\left(\frac{\sigma}{2}\right)_{s's}\cdot\vec{B}(x)$$

$$\simeq -\frac{qF(0)}{m}\left(\frac{\sigma}{2}\right)_{s's}\cdot\vec{B}\,\delta^3(p-p')$$

$$\equiv -\frac{\mu}{j}\, J^{(j)}_{s's}\cdot B\,\delta^3(p-p')$$

So the magnetic moment of a spin-one-half particle is

$$\mu = \frac{g F(0)}{2m} = \frac{g}{2m}\left(1 - G(0)\right).$$

We dropped the $-\frac{i}{2m}(\overrightarrow{p+p'})\,\delta_{ss'}$

term because it has nothing to do with spin or magnetic moments. It is related to $\vec{p}\cdot\vec{A}$ terms in the energy.