Non-abelian Feynman Diagrams

The interaction hamiltonian is

$$V = -g A^a_\mu \bar{\psi} \gamma^\mu t^a \psi + g f^{abc} \partial_\mu A^a_\nu A^{\mu b} A^{\nu c}$$
$$+ \frac{1}{4} g^2 f^{abc} A^b_\mu A^c_\nu f^{ade} A^{d\mu} A^{e\nu}$$

We have

$$\langle 0 | T(\psi_{i\alpha}(x) \bar{\psi}_{j\beta}(y)) | 0 \rangle = \int \frac{d^4k}{(2\pi)^4} \left(\frac{i}{\not{k} - m} \right)_{\alpha\beta} \delta_{ij} e^{-ik(x-y)} .$$

Note that

$$(\not{k} + m)(\not{k} - m) = \not{k}^2 - m^2 = k_\mu k_\nu \gamma^\mu \gamma^\nu - m^2$$
$$= k^2 - m^2$$

So $\dfrac{1}{\not{k} - m} = \dfrac{\not{k} + m}{k^2 - m^2}$ and so

$$\langle 0 | T(\psi_{i\alpha}(x) \bar{\psi}_{j\beta}(y)) | 0 \rangle = \int \frac{d^4p}{(2\pi)^4} \frac{i \delta_{ij} (\not{p} + m) e^{-ip(x-y)}}{p^2 - m^2 + i\epsilon}$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p} + m)\delta_{ij}}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)} .$$

Also

$$\langle 0|T(A^a_\mu(x)\,A^b_\nu(y))|0\rangle = \int \frac{d^4q}{(2\pi)^4}\frac{-i\eta_{\mu\nu}}{q^2+i\epsilon}\delta^{ab}e^{-iq(x-y)}$$

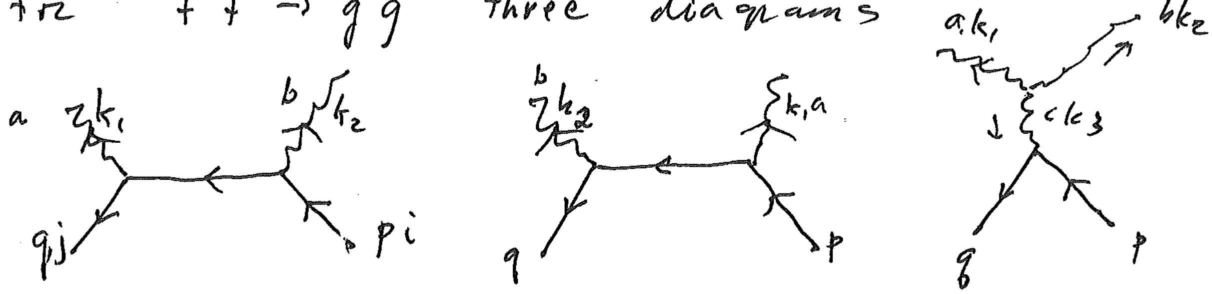
Here

$$\psi_{i\alpha}(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s\left(a(p,s,i)u_\alpha(p,s)e^{-ipx} + b^\dagger(p,s,i)v_\alpha(p,s)e^{ipx}\right)$$

and

$$A^a_\mu(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_k}}\sum_s\left(a(p,s,a)\,\epsilon_\mu(p,s)\,e^{-ipx} + a^\dagger(p,s,a)\,\epsilon^*_\mu(p,s)\,e^{ipx}\right)$$

By the techniques we've been using, we find for $f\bar f \to gg$ three diagrams



The first two as usual give

$$iM_{12} = (ig)^2\,\bar v(q)\Big\{\gamma^\mu t^a\frac{i}{\not p-\not k_2-m}\gamma^\nu t^b + \gamma^\nu t^b\frac{i}{\not k_2-\not q-m}\gamma^\mu t^a\Big\}u(p)\,\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2)$$

The third diagram is new, so we'll look at it more closely

$$\langle k_1 k_2 | T(e^{-i\int V d^4x}) | pq \rangle$$

$$= \langle k_1 k_2 | \frac{(-i)^2}{2} \int T(V_1 V_2) d^4x_1 d^4x_2 \, | pq \rangle$$

$$= \langle k_1 k_2 | -\frac{g^2}{2} \int T\Big[\left(-A^a_\mu \bar\psi \gamma^\mu t^a \psi + f^{abc} \partial_\mu A^a_\nu A^{\mu b} A^{\nu c}\right)_1 d^4x_1$$

$$\left(-A^a_\lambda \bar\psi \gamma^\lambda t^d \psi + f^{abc} \partial_\kappa A^a_\lambda A^{\kappa b} A^{\lambda c}\right)_2 \Big] | pq \rangle d^4x_2$$

$$= +g^2 \langle k_1 k_2 | \int T\left(A^a_\mu \bar\psi \gamma^\mu t^a \psi\right)_1 \left(f \partial_\kappa A^a_\lambda A^{\kappa b} A^{\lambda c}\right)_2 | pq \rangle \, d^4x_1 d^4x_2$$

$$= g^2 \left(\Pi \sqrt{2E's}\right) \langle 0 | a(k_1, s'', a)\, a(k_2, s'', b)$$
$$\times \int T\left(A^a_\mu(x_1) \bar\psi(x_1) \gamma^\mu t^a \psi(x_1) f^{abc} \partial_\kappa A^a_\lambda(x_2) A^{\kappa b}(x_2) A^{\lambda c}(x_2)\right)$$
$$a^\dagger(p, s, \alpha)\, b^\dagger(q, s', \beta) | 0 \rangle$$
$$d^4x_1 d^4x_2$$

$$= g^2 \Pi \sqrt{2E's} \langle 0 | a(k_1, s'', a)\, a(k_2, s''', b)$$
$$\times \int T\left(A^a_\mu(x_1) \frac{\bar v(qs)_\beta}{\sqrt{2E_q}} e^{-iqx_1} \gamma^\mu_{\beta\gamma} t^a_{ji} \frac{u(p,s)}{\sqrt{2E_p}} e^{-ipx_1} f^{abc} \partial_\kappa A^a_\lambda(x_2) A^{\kappa b}(x_2) A^{\lambda c}(x_2)\right) | 0 \rangle$$
$$d^4x_1 d^4x_2$$

Let's look first at

$$\sqrt{2E_{k_1}2E_{k_2}}\langle 0|a(k_1,s'',a)\,a(k_2,s''',b)$$

$$\times\int T\Big[A^c_\mu(x_1)\,e^{-i(q+p)x_1}\,f^{def}\,\partial_\kappa A^d_\lambda(x_2)\,A^{\kappa e}(x_2)\,A^{\lambda f}(x_2)\Big]|0\rangle\,d^4x_1d^4x_2$$

$$=\sqrt{2E_1 2E_2}\int\langle 0|a(k_1,s'',a)\,a(k_2,s''',b)\,e^{-i(q+p)x_1}\,T\Big[A^c_\mu(x_1)$$

$$\times\Big[f^{cef}\,\partial_\kappa A^c_\lambda(x_2)\,A^{\kappa e}(x_2)\,A^{\lambda f}(x_2)$$

$$+f^{dcf}\,\partial_\kappa A^d_\lambda(x_2)\,A^{\kappa c}(x_2)\,A^{\lambda f}(x_2)$$

$$+f^{dec}\,\partial_\kappa A^d_\lambda(x_2)\,A^{\kappa e}(x_2)\,A^{\lambda c}(x_2)\Big]\Big]|0\rangle\,d^4x_1d^4x_2$$

$$=\int e^{-i(q+p)x_1}\Big[\langle 0|T\big(A^c_\mu(x_1)\,\partial_\kappa A^c_\lambda(x_2)\big)|0\rangle f^{cab}\big(\epsilon^{\kappa *}(k_1)\,\epsilon^{\lambda *}(k_2)-\epsilon^{\lambda *}(k_1)\,\epsilon^{\kappa *}(k_2)\big)$$

$$+\langle 0|T\big(A^c_\mu(x_1)\,A^{\kappa c}(x_2)\big)|0\rangle f^{acb}\big(i k_{1\kappa}\,\epsilon^*_\lambda(k_1)\,\epsilon^{\lambda *}(k_2)-i k_{2\kappa}\,\epsilon^*_\lambda(k_2)\,\epsilon^{\lambda *}(k_1)\big)$$

$$+\langle 0|T\big(A^c_\mu(x_1)\,A^{\lambda c}(x_2)\big)|0\rangle f^{abc}\big(i k_{1\kappa}\,\epsilon^*_\lambda(k_1)\,\epsilon^{\kappa *}(k_2)-i k_{2\kappa}\,\epsilon^*_\lambda(k_2)\,\epsilon^{\kappa *}(k_1)\big)$$

$$\Big]\,e^{i(k_1+k_2)x_2}\,d^4x_1\,d^4x_2$$

Now as in QED, $k_\mu \epsilon^{*\mu}(k) = 0$, and $\epsilon_\lambda \epsilon^{\lambda *} = -1$.

So

$$i\mathcal{M}_3 = g^2\, \bar{v}(q,s')\gamma^\mu u(p,s) \int e^{-i(q+p)x_1 + i(k_1+k_2)x_2}\, t^c_{ji}\, d^4x_1\, d^4x_2$$

$$\int \frac{d^4q'}{(2\pi)^4} \frac{-i\, e^{-iq'(x_1-x_2)}}{q'^2+i\epsilon} \Big(i q'_\kappa f^{cab} \left[\epsilon^{\kappa *}(k_1)\epsilon^{\lambda *}(k_2) - \epsilon^{\lambda *}(k_1)\epsilon^{\kappa *}(k_2)\right] \eta_{\mu\lambda}$$

$$+ f^{acb}\, i(k_1-k_2)_\kappa\, \epsilon^*_\lambda(k_1)\epsilon^{\lambda *}(k_2)\, \delta^\kappa_\mu$$

$$+ f^{abc}\, \delta^\lambda_\mu \left(i\, k_{1\kappa}\epsilon^*_\lambda(k_1)\epsilon^{\kappa *}(k_2) - i\, k_{2\kappa}\epsilon^*_\lambda(k_2)\epsilon^{\kappa *}(k_1)\right)\Big)$$

$$= \frac{g^2\, \bar{v}(q,s')\gamma^\mu u(p,s)\, t^c_{ji}}{(q+p)^2} (2\pi)^4\, \delta(k_1+k_2-p-q)$$

$$\times \Big[f^{cab}(-q-p)_\kappa \left[\epsilon^{\kappa *}(k_1)\epsilon^{\lambda *}(k_2) - \epsilon^{\lambda *}(k_1)\epsilon^{\kappa *}(k_2)\right]\eta_{\mu\lambda}$$

$$+ f^{acb}(k_1-k_2)_\kappa\, \epsilon^*_\lambda(k_1)\epsilon^{\lambda *}(k_2)\, \delta^\kappa_\mu$$

$$+ f^{abc}\left[k_{1\kappa}\epsilon^{\kappa *}(k_2)\epsilon^*_\mu(k_1) - k_{2\kappa}\epsilon^{\kappa *}(k_1)\epsilon^*_\mu(k_2)\right]\Big]$$

$$i\mathcal{M}_3 = (2\pi)^4\, \delta(k_1+k_2-p-q)\, g^2\, \frac{\bar{v}(q s')\,\gamma^\mu u(p s)}{(q+p)^2}\, t^c_{ji}$$

$$\times \Big[-f^{abc}(k_1+k_2)_\kappa \left[\epsilon^{\kappa *}(k_1)\,\epsilon^*_\mu(k_2) - \epsilon^*_\mu(k_1)\,\epsilon^{\kappa *}(k_2)\right]$$

$$- f^{abc}(k_1-k_2)_\mu\, \epsilon^*_\lambda(k_1)\,\epsilon^{\lambda *}(k_2)$$

$$+ f^{abc}\left[k_1\cdot\epsilon^*(k_2)\,\epsilon^*_\mu(k_1) - k_2\cdot\epsilon^*(k_1)\,\epsilon^*_\mu(k_2)\right] \Big]$$

$$= (2\pi)^4\, \delta^4(p+q-k_1-k_2)\, g^2\, \frac{\bar{v}(q, s')\,\gamma^\mu u(p s)}{(q+p)^2}\, f^{abc}\, t^c_{ji}$$

$$\times \Big[-2k_2\cdot\epsilon^*(k_1)\,\epsilon^*_\mu(k_2) + 2k_1\cdot\epsilon^*(k_2)\,\epsilon^*_\mu(k_1)$$

$$- (k_1-k_2)_\mu\, \epsilon^*(k_1)\cdot\epsilon^*(k_2) \Big]$$

since $\quad k_1\cdot\epsilon(k_1) = k_2\cdot\epsilon(k_2) = 0,$

whence $\quad k_1\cdot\epsilon^*(k_1) = k_2\cdot\epsilon^*(k_2) = 0.$

We get the same answer by applying the Yang-Mills Feynman rules:

$$i\mathcal{M}_3 = (2\pi)^4 \delta^{(4)}(p+q-k_1-k_2)\, i g\, \bar{v}(q,s')\gamma^\mu u(p,s)\, t^d_{ji}$$

$$\times \frac{-i\eta_{\mu\rho}\,\delta^c_d}{(p+q)^2}\, g f^{abc}\Big[\eta^{\sigma\tau}(-k_1+k_2)^\rho$$

$$+\eta^{\tau\rho}(-k_2-k_1-k_2)^\sigma$$

$$+\eta^{\rho\sigma}(k_1+k_2+k_1)^\tau\Big]\epsilon^*_\sigma(k_1)\epsilon^*_\tau(k_2)$$

$$= (2\pi)^4\delta(p+q-k_1-k_2)\, g^2\, \bar{v}(q,s')\gamma^\mu u(p,s)\, t^c_{ji}\, f^{abc}\frac{1}{(p+q)^2}$$

$$\times\Big[\eta^{\sigma\tau}(k_2-k_1)_\mu - \delta^\tau_\mu(k_1+2k_2)$$

$$+\delta^\sigma_\mu(2k_1+k_2)^\tau\Big]\epsilon^*_\sigma(k_1)\epsilon^*_\tau(k_2)$$

$$= (2\pi)^4\delta^{(4)}(p+q-k_1-k_2)\, g^2\, \bar{v}(q,s')\gamma^\mu u(p,s)\, t^c_{ji}\, f^{abc}$$

$$\times\frac{1}{(p+q)^2}\Big[(k_2-k_1)_\mu\, \epsilon^*(k_1)\cdot\epsilon^*(k_2) - 2k_2\cdot\epsilon^*(k_1)\epsilon^*_\mu(k_2)$$

$$+ 2k_1\cdot\epsilon^*(k_2)\,\epsilon^*_\mu(k_1)\Big].$$