# Anomalous Magnetic Moment of the Electron

We have seen that one can write the matrix element of the current $J^\mu(0)$ as

$$\langle p', s' | J^\mu(0) | p, s\rangle = \frac{iq}{(2\pi)^3} \bar{u}(p', s') \Gamma^\mu(p', p) u(p, s)$$

where

$$\bar{u}(p', s') \Gamma^\mu(p', p) u(p, s) = \bar{u}(p', s') \left[ \gamma^\mu F(q^2) - \frac{i}{2m}(p+p')^\mu G(q^2) \right] u(p, s)$$

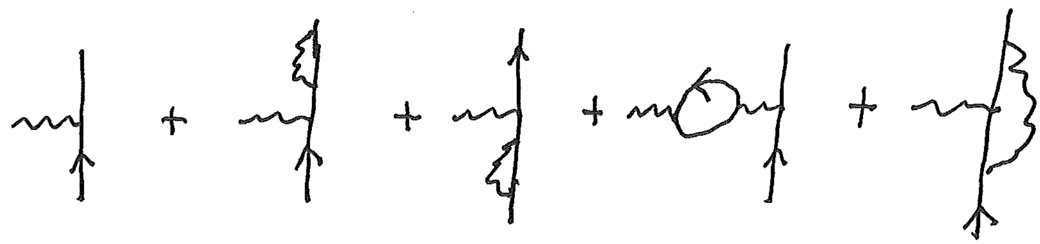
where $q = p - p'$. We also saw that

$$F(0) + G(0) = 1$$

and that the magnetic moment of a spin-1/2 particle of charge $q$ is

$$\mu = \frac{q F(0)}{2m} = \frac{q}{2m}(1 - G(0)).$$

The diagrams are

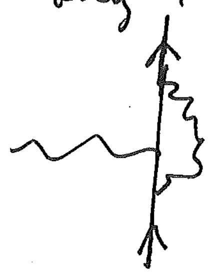


The first diagram gives $F(q^2) = 1$ and $G(q^2) = 0$, which gives $\mu = q/2m$, the normal magnetic moment.

The incoming and outgoing electrons are "on the mass shell" — that is, $p^2 = p'^2 = -m^2$, and so we can ignore the 2d and 3d diagrams by (10.3.30) of SWI, i.e., $\Sigma^*(i m) = 0$.

Also, the 4th diagram makes no contribution to the magnetic moment of the electron because $q^2 = 0$, and so $\Pi(q^2) = \Pi(0) = 0$ by (10.5.19).

So only the 5th diagram



is relevant.

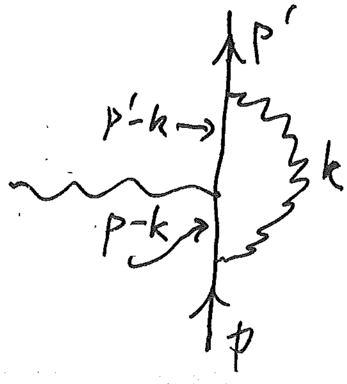
$$\Gamma^\mu(p', p) = \int d^4k \left[e\gamma^\rho (2\pi)^4\right]\left[\frac{-i}{(2\pi)^4}\,\frac{-i(\not{p}'-\not{k})+m}{(p'-k)^2+m^2-i\epsilon}\right]\gamma^\mu$$

$$\times \left[\frac{-i}{(2\pi)^4}\,\frac{-i(\not{p}-\not{k})+m}{(p-k)^2+m^2-i\epsilon}\right]\left[e\gamma_\rho (2\pi)^4\right]$$

$$\times \left[\frac{-i}{(2\pi)^4}\,\frac{1}{k^2-i\epsilon}\right]$$

Here

$q = p - p'$



We use Feynman's trick

$$\frac{1}{ABC} = 2\int_0^1 dx \int_0^x dy \frac{1}{[Ay + B(x-y) + C(1-x)]^3}$$

to write the denominators as

$$\frac{1}{(p'-k)^2+m^2-i\epsilon} \frac{1}{(p-k)^2+m^2-i\epsilon} \frac{1}{k^2-i\epsilon}$$

$$= 2\int_0^1 dx \int_0^x dy \Big[ ((p'-k)^2+m^2-i\epsilon)y + ((p-k)^2+m^2-i\epsilon)(x-y) + (k^2-i\epsilon)(1-x) \Big]^{-3}$$

$$= 2\int_0^1 dx \int_0^x dy \Big[ (k-p'y-p(x-y))^2 + m^2x^2 + q^2 y(x-y) - i\epsilon \Big]^{-3}.$$

Assuming that we've regulated the divergences somehow, we replace $k$ by $k + p'y + p(x-y)$ to get

$$\Gamma^\mu(p',p) = \frac{2ie^2}{(2\pi)^4}\int_0^1 dx \int_0^x dy \int d^4k \, \frac{\gamma^\rho[-i(\not p'(1-y) - \not k - \not p(x-y)) + m]\gamma^\mu \times [-i(\not p(1-x+y) - \not k - \not p' y) + m]\gamma_\rho}{[k^2+m^2x^2+q^2y(x-y) - i\epsilon]^3}.$$

We now Wick rotate letting $k^0 = i k^4$ and integrate over $k_1, k_2, k_3, k_4$ from $-\infty$ to $\infty$. We drop terms odd in $k^\mu$ due to symmetry of

$$k^2 = \sum_{i=1}^{4} (k^i)^2.$$

$d^4 k = i\, d^4 k$. The area of a unit sphere in $d=4$ dimensions is

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} = \frac{2\pi^2}{\Gamma(2)} = 2\pi^2.$$

We get

$$\Gamma^\mu(p',p) = -\frac{4\pi^2 e^2}{(2\pi)^4} \int_0^1 dx \int_0^x dy \int_0^\infty k^3 dk$$

$$\times \frac{\left\{ -\frac{k^2}{4} \gamma^\rho \gamma^\sigma \gamma^\mu \gamma_\sigma \gamma_\rho + \gamma^\rho \left[ -i(\not{p}'(1-y) - \not{p}(x-y)) + m \right] \gamma^\mu \times \left[ -i(\not{p}(1-x+y) - \not{p}' y) + m \right] \gamma_\rho \right\}}{\left[ k^2 + m^2 x^2 + q^2 y (x-y) \right]^3}$$

in which the term with 5 $\gamma$'s came from $\gamma^\rho \not{k} \gamma^\mu \not{k} \gamma_\rho$.

Now we move $\not{p}$ to the right and $\not{p}'$ to the left. For instance,

$$\not{p}\,\gamma_\rho = p^\alpha \gamma_\alpha \gamma_\rho = p^\alpha(\eta_{\alpha\rho} - \gamma_\rho\gamma_\alpha) = p_\rho - \gamma_\rho \not{p}.$$

One then finds, after using $\bar{u}'(i\not{p}'+m) = 0$ and $(i\not{p}+m)u = 0$, that

$$\bar{u}'\,\Gamma^\mu(p',p)u = \frac{-4\pi^2 e^2}{(2\pi)^4}\int_0^1 dx \int_0^x dy \int_0^\infty k^3\,dk$$

$$\frac{\bar{u}'\{\gamma^\mu[-k^2 + 2m^2(x^2-4x+2) + 2q^2(y(x-y)+1-x)] + 4im\,p'^\mu(y-x+xy) + 4im\,p^\mu(x^2-xy-y)\}u}{[k^2+m^2x^2+q^2y(x-y)]^3}.$$

To get $\mu$ quickly, we set $q^2=0$ and $p=p'$. Then, keeping $p'$ to find $G(0)$,

$$\bar{u}\Gamma^\mu(p,p)u = -\frac{e^2}{4\pi^2}\int_0^1 dx\int_0^x dy\int_0^\infty k^3\,dk$$

$$\frac{\bar{u}\{\gamma^\mu[-k^2+2m^2(x^2-4x+2)] + 4im\,\frac{1}{2}(p'^\mu+p^\mu)\,x(x-1)\}u}{[k^2+m^2x^2]^3}$$

Then $G(0)$ is

$$G(0) = \frac{2m}{(-i)}\left(\frac{-e^2}{4\pi^2}\right)\int_0^1 dx \int_0^x dy \int_0^\infty k^3\, dk$$

$$\frac{2im\, x(x-1)}{[k^2 + m^2 x^2]^3}$$

$$= \frac{m^2 e^2}{\pi^2}\int_0^1 dx \int_0^x dy \int_0^\infty dk \; \frac{k^3\, x(x-1)}{[k^2 + m^2 x^2]^3}$$

Let $u = k^2$, $\quad du = 2k\, dk$, so

$$G(0) = -\frac{m^2 e^2}{2\pi^2}\int_0^1 dx \int_0^x dy \int_0^\infty \frac{x(1-x)\, u\, du}{[u + m^2 x^2]^3}$$

$$\int_0^\infty \frac{u\, du}{(u+a^2)^3} = \frac{1}{2a^2} \qquad \text{so}$$

$$G(0) = -\frac{m^2 e^2}{4\pi^2}\int_0^1 dx \int_0^x dy \; \frac{x(1-x)}{m^2 x^2}$$

$$= -\frac{m^2 e^2}{4\pi^2}\int_0^1 dx \; \frac{1-x}{m^2} = -\frac{e^2}{4\pi^2}\left[x - \frac{x^2}{2}\right]_0^1 = -\frac{e^2}{8\pi^2}$$

Thus

$$G(0) = -\frac{e^2}{8\pi^2}$$

and so the magnetic moment of the election to order $e^3$ is

$$\mu = \frac{e}{2m}\left(1 - G(0)\right)$$

$$= \frac{e}{2m}\left(1 + \frac{e^2}{8\pi^2}\right)$$

in natural units where $\hbar = c = 1$ and

$$\alpha = \frac{e^2}{4\pi\hbar c} = \frac{1}{137.036}$$

So

$$\mu = \frac{e}{2m}\left(1 + \frac{\alpha}{2\pi}\right) = \frac{e}{2m}\left(1 + \frac{1}{2\pi(137.036)}\right)$$

$$= \frac{e}{2m}\, 1.001161 .$$ Julian Schwinger 1948

The experimental value is $\frac{e}{2m}\, 1.00115965218111$