Yukawa's Example

Consider the amplitude

\[ G(q_1, \ldots, q_4) = \int d^4x_1 \cdots d^4x_4 e^{i \phi_i x_i} \langle 0 | T(\psi(x_1) \psi(x_2) \psi(x_3) \psi(x_4)) | 0 \rangle \]

Focus on

\[ G(1) = \int d^4x \cdots d^4x e^{i \phi_i x_i} \langle 0 | T(\psi(x_1) \psi(x_2)) | \psi \rangle \]

\[ T(\psi(x_3) \psi(x_4)) | 0 \rangle \langle 0 | T(\psi(x_3) \psi(x_4)) \psi \rangle \]

in which \( | \psi \rangle \) is a \( \pi \)-meson of momentum \( p \).

Then this amplitude will have a pole at

\[ q = q_1 + q_2 = -q_3 - q_4 \quad \text{at} \quad q^2 = -m_\pi^2. \]

\[ m_\pi \approx 140 \text{ MeV}/c^2. \]

\[ G(1) \propto \frac{\delta^4(q_1 + q_2 - q_3 - q_4)}{(q_1 + q_2)^2 + m_\pi^2 - i\epsilon}. \]

The energy transfer \( q_1 + q_2 \) for \( |q_1|^2, |q_2|^2 \ll m_\pi^2\) and \( |q_3|^2, |q_4|^2 \ll m_\pi^2 \) is of the order of \( (q_1^2 + q_2^2)/2m_\pi \) and \( s \).

\[ G(q_1, \ldots, q_4) \sim \frac{s^{\frac{4}{2}}}{(q_1^2 + q_2^2)^2 + m_\pi^2} \]

but since \( q_1 \) is outgoing and \( q_2 \) incoming.
\[ G \sim \frac{\delta}{(p_1 - p_2)^2 + m^2} \]

where the \( p \)'s are the ordinary 3-momenta of the quarks.

Now \((p_1 - p_2)^2 + m^2\) does not vanish, but because \( m^2 \) is the lightest hadron at \( \approx 140 \) MeV, \( G \) is affected by the pole even though its in the unphysical region.

A local potential \( \phi \) was suggested by Yukawa. In the Born approximation, it gives

\[
\int d^3x_1 d^3x_2 d^3x_1' d^3x_2' e^{-i\mathbf{x}_1 \cdot \mathbf{p}_1 - i\mathbf{x}_2 \cdot \mathbf{p}_2 + i\mathbf{x}_1' \cdot \mathbf{p}_1' + i\mathbf{x}_2' \cdot \mathbf{p}_2'} \frac{\delta^3(\mathbf{x}_1 - \mathbf{x}_1')}{4\pi i \mathbf{x}_1 - \mathbf{x}_2} \frac{\delta^3(\mathbf{x}_2 - \mathbf{x}_2')}{4\pi i \mathbf{x}_2 - \mathbf{x}_2'}
\]

\[ = -\frac{(2\pi)^3}{4\pi} \frac{\delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_1' - \mathbf{p}_2')}{(\mathbf{p}_1 - \mathbf{p}_1')^2 + m^2} \]

due to the pion particle.