Wilczek on Anyons

Let's first refer to Weinberg's formula (7.4.23) for the angular-momentum operators

$$J^{ij} = \int d^3x \, \frac{\partial \mathcal{L}}{\partial \dot{\psi}^\ell} \left(-x^i \partial_j \psi^\ell + x^j \partial_i \psi^\ell - i (\mathcal{J}^{ij})^\ell{}_m \psi^m \right)$$

in which the matrices $\mathcal{J}^{ij}$ describe how the fields ψ^m transform under Lorentz transformations

$$\delta \psi^\ell = \frac{i}{2} \omega^{\mu\nu} (\mathcal{J}_{\mu\nu})^\ell{}_m \psi^m .$$

For a covariant vector field

$$(\mathcal{J}_{\rho\sigma})_\kappa{}^\lambda = -i \eta_{\rho\kappa} \delta_\sigma^\lambda + i \eta_{\sigma\kappa} \delta_\rho^\lambda ,$$

but for a massless gauge field, there's also a gauge transformation.

For a nonabelian theory

$$\frac{\partial \mathcal{L}}{\partial(\partial_0 A_{\alpha i})} = -F_\alpha{}^{0i} = \partial_0 A_{\alpha i} - \partial_i A_{\alpha 0} + C_{\alpha\beta\gamma} A_{\beta 0} A_{\gamma i}$$

(15.4.5)

where the C's are the structure constants.

So the J^{ij}'s contain cubic terms in the gauge fields, that is

$$J^{ij} = \int d^3x \left(-F_\alpha{}^{0\ell}\right)\left(-x^i \partial_j A^{\alpha\ell} + x^j \partial_i A^{\alpha\ell} - i\left(J^{ij}\right)^{\ell}_{m} A^{\alpha m}\right)$$

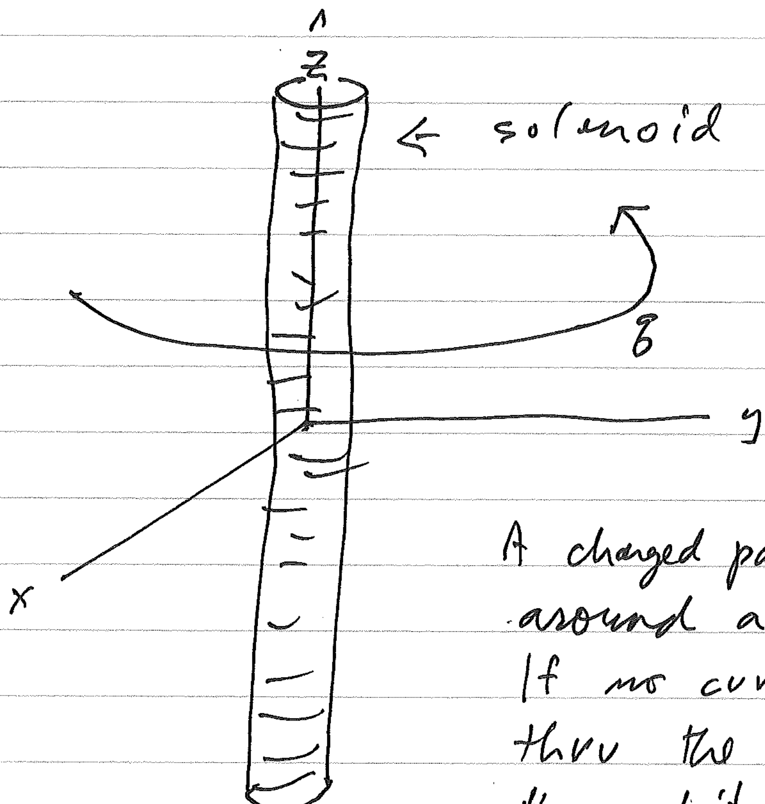
has terms like

$$C_{\alpha\beta\gamma}\, A_{\beta 0}\, A_{\gamma\ell}\, A^{\alpha\ell}.$$

So we expect gauge fields to carry angular momentum in ways that may be surprising. Indeed $L = r \times p \rightarrow L = \vec{r} \times (\vec{p} - q\vec{A})$ at the very least.

Now to Wilczek's 1982 papers:

(A)



A charged particle circles around a solenoid. If no current flows thru the solenoid, then the orbital angular momentum $l_z = -i\partial_\phi$ is quantized as an integer.

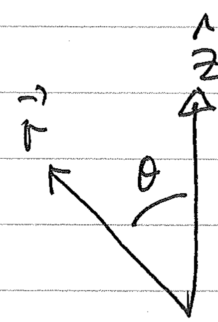
Now slowly turn on the current thru the solenoid.

By Faraday's law, $\nabla \times E = -\dot{B}$, the particle of charge q will feel an electric field

$$\vec{E}(\vec{r}) = - \frac{\hat{z} \times \vec{r}}{2\pi(x^2+y^2)} \dot{\Phi}$$

where Φ is the magnetic flux thru the solenoid. So $\dot{l}_z = (r \times \dot{p})_z$ or

$$\dot{l}_z = (\vec{r} \times q\vec{E})_z$$



$$= -\frac{q}{2\pi} \frac{(\vec{r} \times (\hat{z} \times \vec{r}))_z}{x^2+y^2} \dot{\Phi}$$

$$= -\frac{q}{2\pi} \frac{r^2 \sin^2\theta}{x^2+y^2} \dot{\Phi}$$

$$= -\frac{q}{2\pi} \dot{\Phi}.$$

So Δl_z depends only upon the final flux Φ

$$\Delta l_z = -\frac{q}{2\pi} \Phi.$$

But now

$$l_z = \text{integer} - \frac{q\Phi}{2\pi}.$$

(B) Same physics, different viewpoint:

The mechanical momentum $m\vec{v} = \vec{p} - q\vec{A}$, so

$$l_z = -i\partial_\phi - qA_\phi .$$

The field $A_\phi = \dfrac{\Phi}{2\pi}$ makes flux Φ

inasmuch as

$$\Phi = \int B\cdot ds = \int(\nabla\times A)\cdot ds = \int A\cdot dl = 2\pi A_\phi .$$

$$\left(\int_M F = \int_M dA = \int_{\partial M} A \right)$$

If the wave function is

$$\psi_n \propto e^{in\phi}$$

then n must be an integer for continuity, and so

$$l_z \psi_n = (-i\partial_\phi - qA_\phi)\psi_n$$

$$= \left(-i\partial_\phi - \frac{q\Phi}{2\pi}\right)\psi_n$$

$$= \left(n - \frac{q\Phi}{2\pi}\right)\psi_n .$$

So the two viewpoints (A) and (B) lead to the same surprising conclusion.

(C) Now let's use a singular gauge by making the gauge transformation

$$\vec{A}' = \vec{A} - \nabla\Lambda \quad \text{where } \Lambda = \frac{\Phi\phi}{2\pi}$$

which is singular because Λ is multivalued, aka discontinuous. Now $\psi' = \exp(-iq\Lambda)\psi$ so

$$\psi'(\phi+2\pi) = \exp(-i\Lambda(\phi+2\pi))\,\psi(\phi+2\pi)$$

$$= e^{-iq\frac{\Phi(\phi+2\pi)}{2\pi}}\,\psi(\phi)$$

$$= e^{-iq\Phi}\,\psi'(\phi).$$

Equivalently, $\psi_n'(\phi) = e^{-iq\Lambda}\,\psi_n(\phi)$

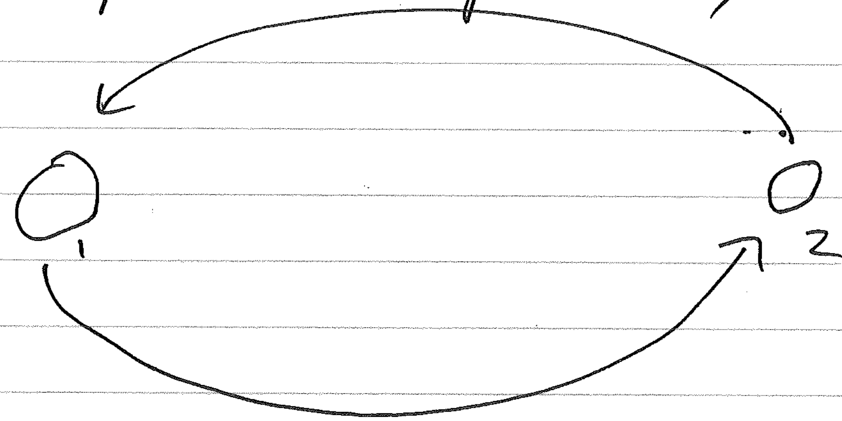
$$= e^{-iq\frac{\Phi\phi}{2\pi}}\,e^{in\phi}$$

$$= e^{i(n - q\Phi/2\pi)\phi}.$$

Now $\vec{A}'_\phi = A_\phi - \partial_\phi\Lambda = \frac{\Phi}{2\pi} - \frac{\Phi}{2\pi} = 0$, so

$l_z = -i\partial_\phi$ and $l_z\psi_n'(\phi) = \left(n - \frac{q\Phi}{2\pi}\right)\psi_n'$.

Now consider two flux-tube-charged-particle composites, and move



one around the other so as to interchange them.

Moving 2 around 1 by π gives an extra factor

$$e^{+iq\int_0^\pi A_\phi d\phi} = e^{+iq\pi\,\Phi/2\pi} = e^{+iq\Phi/2}.$$

But moving 1 around 2 by π also gives $e^{iq\Phi/2}$.

So the total phase is $e^{iq\Phi}$.

So if $q\Phi = \pi$ or $q\Phi = (2m+1)\pi$, then the statistics of the composites are the opposite of what one expects. Equivalently, if $l_z = \text{integer} - q\Phi/2\pi$ is an integer, the statistics are normal, e.g., flux tube plus electron is a fermion. But if $l_z = \text{integer} - q\Phi/2\pi$ is a half-odd integer, then the statistics are inverted.

If $l_z = \text{integer} - \frac{q\Phi}{2\pi}$ is neither an integer nor a half-odd integer, then we have an anyon.