Wilczek on Anyons

Let's first refer to Weinberg’s formula (7.4.23) for the angular-momentum operators

\[ J^{ij} = \int d^3 x \frac{2i}{2\pi^2} ( - i \partial_{x_i} y^I + x (\partial_{x_i} y^I - \frac{1}{m} \hat{J}^I \partial_{y^I} y^J) ) \]

in which the matrices \( \hat{J}^I \) describe how the fields \( y^m \) transform under Lorentz transformation

\[ \delta y^I = \frac{i}{2} \omega^{mn} \left( \hat{J}^n \right)^I_m y^m. \]

For a covariant vector field

\[ (\hat{J}^0)^I_m = -i \gamma^k \delta^I_m \gamma^0 + i \gamma^0 \gamma^k, \]

but for a massless gauge field, there's also a gauge transformation.

For a monopole-like theory

\[ \frac{2i}{(2\pi)^4} = - F_0^{0\gamma} = \partial_\alpha A_\gamma - \partial_\gamma A_\alpha + C_{\alpha\beta\gamma} A_\rho A^{\rho\gamma} \]

\[ (15.4.5) \]

where the C's are the structure constants.

So the \( J^{ij} \)'s contain cubic terms in the gauge fields, that is...
\[ J^{ij} = \int d^3x \left(- F_{a}^{\alpha \beta} \right) \left( -x_i \partial_j A^\alpha + x_j \partial_i A^\alpha - i (j^{ij})^{\alpha \beta} A^\beta \right) \]

has terms like

\[ C \alpha \rho A^\alpha \eta A^\eta \]

So we expect gauge fields to carry angular momentum in ways that may be surprising. Indeed \( L = \vec{r} \times p = \vec{r} \times (\vec{p} - q \vec{A}) \) at the very least.

Now to Wilczek's 1982 papers:

A charged particle circles around a solenoid. If no current flows thru the solenoid, then the orbital angular momentum \( \ell_z = -i \partial \phi \) is quantized as an integer.
Now slowly turn on the current thru the solenoid.

By Faraday's law, \( \nabla \times \mathbf{E} = -\dot{\mathbf{B}} \), the particle of charge \( q \) will feel an electric field

\[
\mathbf{E}^2(r^3) = -\frac{\hat{z} \times \hat{r}}{2\pi(x^2 + y^2)} \Phi
\]

where \( \Phi \) is the magnetic flux thru the solenoid. So \( \ell_2 = (r \times \dot{\mathbf{p}})_z \) or

\[
\ell_2 = (\mathbf{r} \times \mathbf{qE})_z \\
= \frac{-q}{2\pi} \frac{(\hat{r} \times (\hat{z} \times \mathbf{r^3}))_z \Phi}{x^2 + y^2} \\
= \frac{-q}{2\pi} \frac{r^2 \sin^2 \theta}{x^2 + y^2} \Phi \\
= \frac{-q}{2\pi} \dot{\Phi}.
\]

So \( \Delta \Phi_2 \) depends only upon the final flux \( \Phi \)

\[
\Delta \Phi_2 = -\frac{q}{2\pi} \dot{\Phi}.
\]

But now

\[
\ell_2 = \text{inphase} - \frac{q}{2\pi} \Phi.
\]
(b) Same physics, different viewpoint:

The mechanical momentum $\vec{\pi} = \vec{p} - q \vec{A}$, so

$$\vec{l}_z = -i \partial \phi - \frac{1}{2} \vec{A} \phi.$$ 

The field $A \phi = \frac{\Phi}{2\pi}$ makes $\vec{H} \times \vec{F}$

isomorphic as

$$\vec{F} = \int B \cdot ds = \int (\nabla \times A) \cdot ds = \int A \cdot dl = 2\pi A \phi. $$

$$\left( \int F = \int \frac{dA}{\partial n} = \int \frac{A}{\partial m} \right)$$

If the wave function is

$$\psi_n \propto e^{-i m \phi}$$

then $n$ must be an integer for continuity, and so

$$\vec{l}_z \psi_n = (-i \partial \phi - \frac{1}{2} \vec{A} \phi) \psi_n \psi_n$$

$$= (-i \partial \phi - \frac{1}{2} \frac{\Phi}{2\pi} \psi_n \psi_n$$

$$= \left( m - \frac{\Phi}{2\pi} \right) \psi_n.$$
So the two viewpoints (A) and (B) lead to the same surprising conclusion.

(C) Now let's use a singular gauge by making the gauge transformation

\[ A' = A - \nabla \lambda \quad \text{where} \quad \lambda = \frac{\Phi}{2\pi} \]

which is singular because \( \lambda \) is multi-valued, aka discontinuous. Now \( \psi' = \exp(-i\lambda)\psi \)

so

\[ \psi'(\Phi + 2\pi) = \exp(-i\lambda(\Phi + 2\pi))\psi(\Phi + 2\pi) \]

\[ = e^{-i\frac{\Phi}{2\pi}}\psi(\Phi) \]

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Equivalently, \( \psi'(\Phi) = e^{i\Phi/2\pi}\psi(\Phi) \)

\[ = e^{i(n - \frac{\Phi}{2\pi})\Phi} \]

\[ = e. \]

Now \( A'q = Aq - dq \lambda = \frac{\Phi}{2\pi} - \frac{\Phi}{2\pi} = 0 \), so

\[ L^2 = -i\hbar d\Phi \quad \text{and} \quad L^2 \psi'(\Phi) = (n - \frac{\Phi}{2\pi})\psi'. \]
Now consider two flux-tube-charged-particle composites, and move

\[ \Phi \]

around the other so as to interchange them.

Moving 2 around 1 by \( \pi \) gives an extra factor \( \pi \)

\[ +i \beta \int A d\ell + i q \pi \phi/2 \pi + i q \Phi/2 \]

\[ e = e_i \]

But moving 1 around 2 by \( \pi \) also gives \( e_i \).

So the total phase is \( e_i \).

So if \( q \Phi = \pi \) or \( q \Phi = (2m+1) \pi \),
then the statistics of the composites are the opposite of what one expects. Equivalently, if \( l_2 \) = integer - \( q \Phi/2 \pi \) is an integer, the statistics are normal, e.g., flux tube plus electron is a fermion.

But if \( l_2 = \) integer - \( q \Phi/2 \pi \) is a half-odd integer, then the statistics are inverted.
If \( l_2 \) is an integer, \( 0 < \theta < \frac{2\pi}{l_2} \) is neither an integer nor a half-odd integer, then we have an anomaly.