Warning: these notes have been translated from the normal metric to the Peskin metric ($+, -, -, -$).

Example: Feynman's propagator for a spinless quantum field $\phi(x)$ of mass $m$ is

$$\Delta_F(x) = \int \frac{\exp(-ikx)}{-k^2 + m^2 - i\epsilon} \frac{d^4k}{(2\pi)^4}$$  \hspace{1cm} (0.1)

where

$$kx \equiv -k \cdot x + k^0 x^0$$  \hspace{1cm} (0.2)

$x^0 = ct$, and all physical quantities are in natural units ($c = \hbar = 1$). The tiny imaginary term $-i\epsilon$ makes $\Delta_F(x-y)$ proportional to the mean-value in the vacuum state $|0\rangle$ of the time-ordered product

$$\mathcal{T}\{\phi(x)\phi(y)\} \equiv \theta(x^0 - y^0)\phi(x)\phi(y) + \theta(y^0 - x^0)\phi(y)\phi(x)$$  \hspace{1cm} (0.3)

of the fields $\phi(x)$ and $\phi(y)$ in which $\theta(a) = (a + |a|)/(2|a|)$ is the Heaviside step function. The exact formula is

$$\langle 0|\mathcal{T}\{\phi(x)\phi(y)\}|0\rangle = -i \Delta_F(x-y).$$  \hspace{1cm} (0.4)

P&S's $D_F$ is $\langle 0|\mathcal{T}\{\phi(x)\phi(y)\}|0\rangle$

$$D_F(x-y) = -i \Delta_F(x-y) = \langle 0|\mathcal{T}\{\phi(x)\phi(y)\}|0\rangle = \int e^{-ikx} \frac{i}{k^2 - m^2 + i\epsilon} \frac{d^4k}{(2\pi)^4}.$$  \hspace{1cm} (0.5)

Example—The Feynman Propagator: Adding $\pm i\epsilon$ to the denominator of a pole term of an integral formula for a function $f(x)$ can slightly shift the pole into the upper or lower half plane, causing the pole to contribute if a ghost contour goes around the UHP or the LHP. The choice of ghost contour often is influenced by the argument $x$ of the function $f(x)$. Such $i\epsilon$'s impose boundary conditions on Green's functions.

The Feynman propagator $\Delta_F(x)$ is a Green's function for the Klein-Gordon differential operator (Weinberg, 1995, pp. 274–280)

$$(\Box + m^2) \Delta_F(x) = \delta^4(x)$$  \hspace{1cm} (0.6)

in which $x = (x^0, \vec{x})$ and

$$\Box = \frac{\partial^2}{\partial t^2} - \Delta = \frac{\partial^2}{\partial (x^0)^2} - \Delta$$  \hspace{1cm} (0.7)

is the four-dimensional version of the laplacian $\Delta \equiv \nabla \cdot \nabla$. Here $\delta^4(x)$ is the
four-dimensional version of Dirac’s delta function
\[ \delta^4(x) = \int \frac{d^4q}{(2\pi)^4} \exp[\pm i(q \cdot x - q^0 x^0)] = \int \frac{d^4q}{(2\pi)^4} e^{\pm iqx} \] (0.8)
in which \( qx = q^0 x^0 - q \cdot x \) is the Lorentz-invariant inner product of the 4-vectors \( q \) and \( x \). There are many Green’s functions that satisfy Eq. (0.6).

Feynman’s propagator \( \Delta_F(x) \) is the one that satisfies certain boundary conditions which will become evident when we analyze the effect of its \( i\epsilon \)
\[ \Delta_F(x) = \int \frac{d^4q}{(2\pi)^4} \frac{\exp(-iqx)}{-q^2 + m^2 - i\epsilon}. \] (0.9)
The quantity \( E_q = \sqrt{q^2 + m^2} \) is the energy of a particle of mass \( m \) and momentum \( q \) in natural units with the speed of light \( c = 1 \). Using this abbreviation and setting \( \epsilon' = \epsilon/(2E_q) \), we may write the denominator as
\[ -q^2 + m^2 - i\epsilon = q \cdot q - (q^0)^2 + m^2 - i\epsilon = (E_q - i\epsilon' - q^0) (E_q - i\epsilon' + q^0) + \epsilon'^2 \] (0.10)
in which \( \epsilon'^2 \) is negligible. We now drop the prime on the \( \epsilon \) and do the \( q^0 \) integral
\[ I(q) = -\int_{-\infty}^{\infty} \frac{dq^0}{2\pi} e^{-iq^0 x^0} \frac{1}{[q^0 - (E_q - i\epsilon)][q^0 - (-E_q + i\epsilon)]}. \] (0.11)
The function
\[ f(q^0) = e^{-iq^0 x^0} \frac{1}{[q^0 - (E_q - i\epsilon)][q^0 - (-E_q + i\epsilon)]} \] (0.12)
has poles at \( E_q - i\epsilon \) and at \( -E_q + i\epsilon \), as shown in Fig. 0.1. If \( x^0 > 0 \), then we can add a ghost contour that goes cw around the LHP, and we get
\[ I(q) = ie^{-iE_q x^0} \frac{1}{2E_q} x^0 > 0. \] (0.13)
If \( x^0 < 0 \), we add a ghost contour that goes ccw around the UHP, and we get
\[ I(q) = ie^{iE_q x^0} \frac{1}{2E_q} x^0 < 0. \] (0.14)
Using Heaviside’s step function
\[ \theta(x) = \frac{x + |x|}{2}, \] (0.15)
we may combine the last two equations into
\[ -iI(q) = \frac{1}{2E_q} \left[ \theta(x^0) e^{-iE_q x^0} + \theta(-x^0) e^{iE_q x^0} \right]. \] (0.16)
In Eq. (0.12), the function \( f(q^0) \) has poles at \( \pm (E_q - i \epsilon) \), and the function \( \exp(-i q^0 x^0) \) is exponentially suppressed in the LHP if \( x^0 > 0 \) and in the UHP if \( x^0 < 0 \). So we can add a ghost contour in the LHP if \( x^0 > 0 \) and in the UHP if \( x^0 < 0 \).

In terms of the Lorentz-invariant function

\[
\Delta_+(x) = \frac{1}{(2\pi)^3} \int \frac{d^3q}{2E_q} \exp[i(q \cdot x - E_q x^0)]
\]  \hspace{1cm} (0.17)

and with a factor of \(-i\), the Feynman propagator is

\[
-i \Delta_F(x) = \theta(x^0) \Delta_+(x) + \theta(-x^0) \Delta_+(x, -x^0).
\]  \hspace{1cm} (0.18)
But the integral (0.17) defining $\Delta_+(x)$ is insensitive to the sign of $q$, and so

$$\Delta_+(-x) = \frac{1}{(2\pi)^3} \int \frac{d^3q}{2E_q} \exp[i(-q \cdot x + E_q x^0)]$$

(0.19)

$$= \frac{1}{(2\pi)^3} \int \frac{d^3q}{2E_q} \exp[i(q \cdot x + E_q x^0)] = \Delta_+(x, -x^0).$$

Thus we arrive at the standard form of the Feynman propagator

$$-i\Delta_F(x) = \theta(x^0) \Delta_+(x) + \theta(-x^0) \Delta_+(-x).$$

(0.20)

The Lorentz-invariant function $\Delta_+(x-y)$ is the commutator of the positive-frequency part

$$\phi^+(x) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2p^0}} \exp[i(p \cdot x - p^0 x^0)] a(p)$$

(0.21)

of a scalar field $\phi = \phi^+ + \phi^-$ with its negative-frequency part

$$\phi^-(y) = \int \frac{d^3q}{(2\pi)^3 \sqrt{2q^0}} \exp[-i(q \cdot y - q^0 y^0)] a^\dagger(q)$$

(0.22)

where $p^0 = E_p = \sqrt{p^2 + m^2}$ and $q^0 = E_q$. For since the annihilation operators $a(q)$ and the creation operators $a^\dagger(p)$ satisfy the commutation relation

$$[a(q), a^\dagger(p)] = (2\pi)^3 \delta^3(q - p)$$

(0.23)

we have

$$[\phi^+(x), \phi^-(y)] = \int \frac{d^3p d^3q}{(2\pi)^6 \sqrt{2q^0 2p^0}} e^{-ipx+iqy} [a(p), a^\dagger(q)]$$

$$= \int \frac{d^3p}{(2\pi)^3 2p^0} e^{-ip(x-y)} = \Delta_+(x-y)$$

(0.24)

in which $px = p^0 x^0 - p \cdot x$, etc.

Incidentally, at points $x$ that are space-like

$$x^2 = (x^0)^2 - x^2 \equiv -r^2 < 0$$

(0.25)

the Lorentz-invariant function $\Delta_+(x)$ depends only upon $r = +\sqrt{-x^2}$ and has the value (Weinberg, 1995, p. 202)

$$\Delta_+(x) = \frac{m}{4\pi^2 r} K_1(mr)$$

(0.26)
in which the Hankel function $K_1$ is

$$K_1(z) = -\frac{\pi}{2} \left[ J_1(iz) + iN_1(iz) \right] = \frac{1}{z} + \frac{z}{2j+2} \left[ \ln \left( \frac{z}{2} \right) + \gamma - \frac{1}{2j+2} \right] + \ldots$$

(0.27)

where $J_1$ is the first Bessel function, $N_1$ is the first Neumann function, and $\gamma = 0.57721\ldots$ is the Euler-Mascheroni constant.

The Feynman propagator arises most simply as the mean value in the vacuum of the **time-ordered product** of the fields $\phi(x)$ and $\phi(y)$

$$\mathcal{T} \{ \phi(x)\phi(y) \} \equiv \theta(x^0 - y^0)\phi(x)\phi(y) + \theta(y^0 - x^0)\phi(y)\phi(x).$$

(0.28)

Since the operators $a(p)$ and $a^\dagger(p)$ respectively annihilate the vacuum ket $a(p)\vert 0 \rangle = 0$ and bra $\langle 0 \vert a^\dagger(p) = 0$, the same is true of the positive- and negative-frequency parts of the field: $\phi^+(z)\vert 0 \rangle = 0$ and $\langle 0 \vert \phi^-(z) = 0$. Thus, the mean value in the vacuum of the time-ordered product is proportional to the Feynman propagator $-i\Delta_F(x - y)$

$$\langle 0 \vert \mathcal{T} \{ \phi(x)\phi(y) \} \vert 0 \rangle = \langle 0 \vert \theta(x^0 - y^0)\phi(x)\phi(y) + \theta(y^0 - x^0)\phi(y)\phi(x) \vert 0 \rangle$$

$$= \langle 0 \vert \theta(x^0 - y^0)\phi^+(x)\phi^-(y) + \theta(y^0 - x^0)\phi^+(y)\phi^-(x) \vert 0 \rangle$$

$$= \langle 0 \vert \theta(x^0 - y^0)[\phi^+(x),\phi^-(y)] + \theta(y^0 - x^0)[\phi^+(y),\phi^-(x)] \vert 0 \rangle$$

$$= \theta(x^0 - y^0)\Delta_+(x - y) + \theta(y^0 - x^0)\Delta_+(y - x)$$

$$= -i\Delta_F(x - y)$$

(0.29)

in the last step of which we used (0.20). Feynman put $i\epsilon$ in the denominator of the Fourier transform of his propagator to get this result.