ϵ-terms which modify the Dirac propagators

$$\langle 0|\mathcal{T}\left[\bar{\psi}(x_1)\dots\psi(x_{2n})\right]|0\rangle = \frac{\displaystyle\int \bar{\chi}(x_1)\dots\chi(x_{2n})\, e^{iS_0[\chi,\epsilon]}\, D\chi^* D\chi}{\displaystyle\int e^{iS_0[\chi,\epsilon]}\, D\chi^* D\chi}. \tag{17.227}$$

We now as in (17.144) introduce a Grassmann external current $\zeta(x)$ and define a fermionic analog of $Z_0[j]$

$$Z_0[\zeta] \equiv \langle 0|\,\mathcal{T}\left[e^{i\int \bar{\zeta}\psi+\bar{\psi}\zeta\, d^4x}\right]|0\rangle = \frac{\displaystyle\int e^{i\int \bar{\zeta}\chi+\bar{\chi}\zeta\, d^4x} e^{iS_0[\chi,\epsilon]} D\chi^* D\chi}{\displaystyle\int e^{iS_0[\chi,\epsilon]} D\chi^* D\chi}. \tag{17.228}$$

17.14 Application to Non-Abelian Gauge Theories

The action of a (fairly) generic non-abelian gauge theory is

$$S = \int -\frac{1}{4} F_{b\mu\nu} F_b^{\mu\nu} - \bar{\psi}\left(\gamma^\mu D_\mu + m\right)\psi\; d^4x \tag{17.229}$$

in which the Maxwell field is

$$F_{b\mu\nu} \equiv \partial_\mu A_{b\nu} - \partial_\nu A_{b\mu} + g\, f_{bcd}\, A_{c\mu}\, A_{d\nu} \tag{17.230}$$

and the covariant derivative is

$$D_\mu \psi \equiv \partial_\mu \psi - ig\, t_b\, A_{b\mu}\, \psi. \tag{17.231}$$

Here γ^μ is a gamma matrix (9.267), g is a coupling constant, f_{bcd} is a structure constant (9.62), and t_b is a generator (9.56) of the Lie algebra (9.15) of the gauge group.

One may show (Weinberg, 1996, pp. 14–18) that the analog of equation (17.172) for quantum electrodynamics is

$$\langle \Omega|\mathcal{T}\left[\mathcal{O}_1\dots\mathcal{O}_n\right]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1\dots\mathcal{O}_n\, e^{iS}\, \delta[A_{b3}]\, DA\, D\psi}{\displaystyle\int e^{iS}\, \delta[A_{b3}]\, DA\, D\psi} \tag{17.232}$$

in which the functional delta-function

$$\delta[A_{b3}] \equiv \prod_{x,b} \delta(A_{b3}(x)) \tag{17.233}$$

enforces the axial-gauge condition, and $D\psi$ stands for $D\psi^* D\psi$.

Initially, physicists had trouble computing non-abelian amplitudes beyond the lowest order of perturbation theory. Then DeWitt showed how to compute to second order (DeWitt, 1967), and Faddeev and Popov, using path integrals, showed how to compute to all orders (Faddeev and Popov, 1967).

17.15 The Faddeev-Popov Trick

The path-integral tricks of Faddeev and Popov are described in (Weinberg, 1996, pp. 19–27). We start with some gauge-fixing functions $f_b(x)$ which depend upon a set of non-abelian gauge fields $A^b_\mu(x)$. One might have $f_b(x) = A^3_b(x)$ in an axial gauge or $f_b(x) = i\partial_\mu A^\mu_b(x)$ in a Lorentz-invariant gauge.

Under an infinitesimal gauge transformation

$$A^\lambda_{b\mu} = A_{b\mu} + \partial_\mu \lambda_b + g\, f_{bcd}\, \lambda_d\, A_{c\mu} \tag{17.234}$$

the gauge fields change, and so the gauge-fixing functions $f_b(x)$, which depend upon them, also change. The jacobian J of that change is

$$J = \det \left(\frac{\delta f^\lambda_b(x)}{\delta \lambda_c(y)} \right) \Bigg|_{\lambda=0} \equiv \frac{Df^\lambda}{D\lambda} \Bigg|_{\lambda=0} \tag{17.235}$$

which typically involves the delta-function $\delta^4(x-y)$.

Let $B[f]$ denote any functional of the gauge-fixing functions $f_b(x)$ such as

$$B[f] = \prod_{x,b} \delta(f_b(x)) = \prod_{x,b} \delta(A^3_b(x)) \tag{17.236}$$

in an axial gauge, or

$$B[f] = \exp\left[\frac{i}{2} \int (f_b(x))^2 \, d^4x \right] = \exp\left[-\frac{i}{2} \int \left(\partial_\mu A^\mu_b(x) \right)^2 \, d^4x \right] \tag{17.237}$$

in a Lorentz-invariant gauge.

Now let's consider a functional integral like (17.232)

$$\langle \Omega | \mathcal{T} \left[\mathcal{O}_1 \ldots \mathcal{O}_n \right] | \Omega \rangle = \frac{\displaystyle\int \mathcal{O}_1 \ldots \mathcal{O}_n \, e^{iS} \, B[f] \, J \, DA \, D\psi}{\displaystyle\int e^{iS} \, B[f] \, J \, DA \, D\psi} \tag{17.238}$$

in which the operators $\mathcal{O}_k$, the action functional $S[A]$ and the differentials DA and $D\psi$ are gauge invariant. The axial-gauge formula (17.232) is a simple example in which $B[f] = \delta[A_{b3}]$ enforces the axial-gauge condition $A_{b3}(x) = 0$ and the determinant $J = \det\left(\delta_{bc} \partial_\mu \delta(x-y)\right)$ is a constant, which cancels.

If we translate the gauge fields by a gauge transformation Λ, then the ratio (17.244) does not change

$$\langle\Omega|\mathcal{T}\left[\mathcal{O}_1\dots\mathcal{O}_n\right]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1^\Lambda\dots\mathcal{O}_n^\Lambda\, e^{iS^\Lambda}\, B[f^\Lambda]\, J^\Lambda\, DA^\Lambda\, D\psi^\Lambda}{\displaystyle\int e^{iS^\Lambda}\, B[f^\Lambda]\, J^\Lambda\, DA^\Lambda\, D\psi^\Lambda} \tag{17.239}$$

any more than $\int f(y)\,dy$ is different from $\int f(x)\,dx$. Since the operators $\mathcal{O}_k$, the action functional $S[A]$, and the differentials DA and $D\psi$ are gauge invariant, most of the Λ-dependence goes away

$$\langle\Omega|\mathcal{T}\left[\mathcal{O}_1\dots\mathcal{O}_n\right]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1\dots\mathcal{O}_n\, e^{iS}\, B[f^\Lambda]\, J^\Lambda\, DA\, D\psi}{\displaystyle\int e^{iS}\, B[f^\Lambda]\, J^\Lambda\, DA\, D\psi}. \tag{17.240}$$

If $\Lambda\lambda$ is the gauge transformation Λ followed by the gauge transformation λ, then the jacobian J^Λ is a determinant of a product of matrices which is the product of their determinants

$$\begin{aligned} J^\Lambda &= \det\left(\frac{\delta f_b^{\Lambda\lambda}(x)}{\delta\lambda_c(y)}\right)\Big|_{\lambda=0} = \det\left(\int \frac{\delta f_b^{\Lambda\lambda}(x)}{\delta\Lambda\lambda_d(z)}\frac{\delta\Lambda\lambda_d(z)}{\delta\lambda_c(y)}\, d^4z\right)\Big|_{\lambda=0} \\ &= \det\left(\frac{\delta f_b^{\Lambda\lambda}(x)}{\delta\Lambda\lambda_d(z)}\right)\Big|_{\lambda=0} \det\left(\frac{\delta\Lambda\lambda_d(z)}{\delta\lambda_c(y)}\right)\Big|_{\lambda=0} \\ &= \det\left(\frac{\delta f_b^{\Lambda}(x)}{\delta\Lambda_d(z)}\right) \det\left(\frac{\delta\Lambda\lambda_d(z)}{\delta\lambda_c(y)}\right)\Big|_{\lambda=0} \equiv \frac{Df^\Lambda}{D\Lambda}\frac{D\Lambda\lambda}{D\lambda}\Big|_{\lambda=0}. \end{aligned} \tag{17.241}$$

Now we integrate over the gauge transformation Λ with weight function $\rho(\Lambda) = (D\Lambda\lambda/D\lambda|_{\lambda=0})^{-1}$ and find, since the ratio (17.240 is Λ-independent

$$\begin{aligned} \langle\Omega|\mathcal{T}\left[\mathcal{O}_1\dots\mathcal{O}_n\right]|\Omega\rangle &= \frac{\displaystyle\int \mathcal{O}_1\dots\mathcal{O}_n\, e^{iS}\, B[f^\Lambda]\,\frac{Df^\Lambda}{D\Lambda}\, D\Lambda\, DA\, D\psi}{\displaystyle\int e^{iS}\, B[f^\Lambda]\,\frac{Df^\Lambda}{D\Lambda}\, D\Lambda\, DA\, D\psi} \\ &= \frac{\displaystyle\int \mathcal{O}_1\dots\mathcal{O}_n\, e^{iS}\, B[f^\Lambda]\, Df^\Lambda\, DA\, D\psi}{\displaystyle\int e^{iS}\, B[f^\Lambda]\, Df^\Lambda\, DA\, D\psi} \\ &= \frac{\displaystyle\int \mathcal{O}_1\dots\mathcal{O}_n\, e^{iS}\, DA\, D\psi}{\displaystyle\int e^{iS}\, DA\, D\psi}. \end{aligned} \tag{17.242}$$

Thus, the mean-value in the vacuum of a time-ordered product of gauge-invariant operators is a ratio of path integrals over all gauge fields without any gauge fixing. No matter what gauge condition f or gauge-fixing functional $B[f]$ we use, the resulting gauge-fixed ratio (17.244) is equal to the ratio (17.242) of path integrals over all gauge fields without any gauge fixing. All gauge-fixed ratios (17.244) give the same time-ordered products, and so we can use whatever gauge condition f or gauge-fixing functional $B[f]$ is most convenient.

The analogous formula for the euclidean time-ordered product is

$$\langle\Omega|\mathcal{T}_e\left[\mathcal{O}_1\ldots\mathcal{O}_n\right]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1\ldots\mathcal{O}_n\, e^{-S_e}\, DA\, D\psi}{\displaystyle\int e^{-S_e}\, DA\, D\psi} \tag{17.243}$$

where the euclidean action S_e is the space-time integral of the energy density. This formula is the basis for lattice gauge theory.

The path-integral formulas (17.176 & 17.243) derived for quantum electrodynamics therefore also apply to non-abelian gauge theories.

17.16 Ghosts

Faddeev and Popov were interested in showing how to do perturbative computations in which one does fix the gauge. To continue our description of their tricks, we return to gauge-fixed expression (17.232) for the time-ordered product

$$\langle\Omega|\mathcal{T}\left[\mathcal{O}_1\ldots\mathcal{O}_n\right]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1\ldots\mathcal{O}_n\, e^{iS}\, B[f]\, J\, DA\, D\psi}{\displaystyle\int e^{iS}\, B[f]\, J\, DA\, D\psi} \tag{17.244}$$

set

$$f_b(x) = i\partial_\mu A^\mu_b(x) \tag{17.245}$$

and use (17.237) as the gauge-fixing functional $B[f]$

$$B[f] = \exp\left[\frac{i}{2}\int \left(f_b(x)\right)^2\, d^4x\right] = \exp\left[-\frac{i}{2}\int \left(\partial_\mu A^\mu_b(x)\right)^2\, d^4x\right]. \tag{17.246}$$

This functional adds to the action density the term $-(\partial_\mu A^\mu_b)^2/2$ which leads to a gauge-field propagator like the photon's (17.179)

$$\langle 0|\mathcal{T}\left[A^b_\mu(x)A^c_\nu(y)\right]|0\rangle = -\,i\triangle_{\mu\nu}(x-y) = -\,i\int \frac{\eta_{\mu\nu}\delta_{bc}}{q^2 - i\epsilon}\, e^{iq\cdot(x-y)}\,\frac{d^4q}{(2\pi)^4}. \tag{17.247}$$

What about the determinant J? Under an infinitesimal gauge transformation, the gauge field becomes

$$A^\lambda_{b\mu} = A_{b\mu} + \partial_\mu\lambda_b + g\, f_{bcd}\,\lambda_d\, A_{c\mu} \tag{17.248}$$

and so f^λ_b is

$$f^\lambda_b = i\partial^\mu A^\lambda_{b\mu} = i\partial^\mu\left(A_{b\mu} + \partial_\mu\lambda_b + g\, f_{bdc}\,\lambda_c\, A_{d\mu}\right). \tag{17.249}$$

The jacobian J then is the determinant (17.235) of the matrix

$$\left(\frac{\delta f_b(x)}{\delta\lambda_c(y)}\right)\Bigg|_{\lambda=0} = i\delta_{bc}\,\Box\,\delta^4(x-y) + ig\, f_{bdc}\,\frac{\partial}{\partial x^\mu}\left[A^\mu_d(x)\delta^4(x-y)\right] \tag{17.250}$$

that is

$$J = \det\left(i\delta_{bc}\,\Box\,\delta^4(x-y) + ig\, f_{bdc}\,\frac{\partial}{\partial x^\mu}\left[A^\mu_d(x)\delta^4(x-y)\right]\right). \tag{17.251}$$

But we've seen (17.208) that a determinant can be written as a fermionic path integral

$$\det A = \int e^{-\theta^\dagger A\,\theta}\,\prod_{k=1}^n d\theta^*_k d\theta_k. \tag{17.252}$$

Thus we can write the jacobian J as

$$J = \int \exp\left(-\,i\omega^*_b\,\Box\,\omega_b + i\partial_\mu\omega^*_b\, g\, f_{bdc}\, A^\mu_d\omega_c\, d^4x\right) D\omega^* D\omega \tag{17.253}$$

which contributes the terms $-\,\partial_\mu\omega^*_b\partial^\mu\omega_b$ and $\partial_\mu\omega^*_b\, g\, f_{bdc}\, A^\mu_d\omega_c$ to the action density.

Thus, we can do perturbation theory by using the modified action density

$$\mathcal{L}' = -\,\tfrac{1}{4}F_{b\mu\nu}F^{\mu\nu}_b - \tfrac{1}{2}\left(\partial_\mu A^\mu_b\right)^2 - \partial_\mu\omega^*_b\partial^\mu\omega_b + \partial_\mu\omega^*_b\, g\, f_{bdc}\, A^\mu_d\omega_c - \bar\psi\left(\not{D} + m\right)\psi \tag{17.254}$$

in which $\not{D} \equiv \gamma^\mu D_\mu = \gamma^\mu(\partial_\mu - igf_{bcd}A_{b\mu})$. The ghost field ω is a mathematical device, not a physical field describing real particles, which would be spinless fermions violating the spin-statistics theorem (example 9.3).