Quantum Hall Fluids

Consider a single spinless particle of charge $e$ in a plane through which a magnetic field passes perpendicularly

$$\left[ (d_x - i e A_x)^2 + (d_y - i e A_y)^2 \right] \psi = 2m e B \psi .$$

This is a problem in (2+1) dimensions. Landau solved this problem. The energy levels are

$$E_n = (n + \frac{1}{2}) \frac{eB}{m} \quad \text{for } n = 0, 1, 2, \ldots .$$

If $A$ is the area, then the degeneracy is the same for each $n$:

$$d = \frac{eBA}{\hbar} = \frac{eBA}{2\pi \hbar} = \frac{eBA}{2\pi} .$$

The jump in energy $E_{n+1} - E_n = \frac{eB}{m} .

The filling factor $\nu$ is

$$\nu = \frac{N_e}{(eBA/2\pi)} \quad \text{where } N_e \text{ is the number of electrons. For } \nu < 1, \text{ one can add } e \text{'s without needing to go to the next level. But for integral values of } \nu, \text{ all the levels are full or empty. This is the integer}$$
Hall effect: fluids with integral filling factors are incompressible.

An electric field $E_y$ produces a current

$$ J_x = \sigma_{xy} E_y $$

with

$$ \sigma_{xy} = \nu e^2/2\pi = \frac{e^2 N_e}{2\pi e B A} = \frac{e N_e}{B A} $$

But due to impurities

$$ \sigma_{xy} \bigg|_{B=0} = \frac{e}{B A} $$

**Fractional Hall Effect**

Amazingly, a Hall fluid is also incompressible when

$$ \nu = \frac{1}{2m+1} = \frac{1}{3}, \frac{1}{5}, \frac{1}{7} \ldots $$

The number of this quanta is

$$ N \phi = \frac{e B A}{2\pi} $$

Now

$$ \frac{N \phi}{N_e} = \frac{e B A}{2\pi e B A \nu} = \frac{1}{\nu} $$
So the quantum Hall fluid is incompressible when

\[ \frac{N\phi}{Ne} = \frac{1}{2m+1}. \]

That is, when there are \(2m+1\) flux quanta for each electron.

\[ e^{-i(\pi + iq\Phi/2)} = e^{i\frac{e\phi}{2}} \]

Phase change is \( e^{-i\frac{e(2m+1)\Phi}{2m}} = (-1)^m e^{-\frac{i(e/2)(2m+1)}{2}} \)

\[ = (-1)^m e^{-\frac{i(e/2)(2m+1)}{2}} = (-1)^m e^{i\frac{e\Phi}{2}} = +1. \]

So the electrons become bosons and condense.

Kivelson, Hanssen, and Zhang used

\[ \hat{L} = \psi^{\dagger} \left[ i(D_0 - i e A_0) \psi^\dagger + \frac{1}{2m} \psi^{\dagger} (D_i - i e A_i)^2 \psi \right] \]

\[ + V(\psi^{\dagger} \psi). \quad (\ast) \]
We saw that with a Chern-Simons term added to $\mathcal{L}_0$ so that

$$\mathcal{L} = \mathcal{L}_0 + \gamma E_{ij} dA_{ij} + a_n \mathbf{j}^k$$

we can change statistics by $\Theta = 1/4 \pi$.

The effective field theory of a Hall fluid—that is, the theory that works at large distances—is a Chern-Simons theory.

1) Electrons in a plane

From a (2+1)-D system.

2) The electric current $j_\mu$ is conserved $\partial^\mu j_\mu = 0$

3) We want to use a local action density, a polynomial in fields and their derivatives at one point integrated over the point.

4) We are interested in big distances and long times—low $\hbar$ and long $\tau$.

3) $\mathbf{B}$ breaks $\mathbb{P}$ and $\mathbb{T}$.

The coupling constant $g$
of a term in $d$ dimensions

$$\int d^d x \, g_{mnp} \, J^m$$

varies with $L$ — the stretch — as

$$d - n(d-2)/2 - p$$

$$q_{mnp}(L) = L^{d-n(d-2)/2-p} g_{mnp}.$$  

(See section 18.3 of my online notes.)

So, what terms can we have in our long-range theory of a Hulth fluid?

Well, $a^m g_m$ is ruled out by gauge invariance.

The Chern-Simons term $g_{mnp} \varepsilon^{mnp} \propto d \wedge \varphi$

goes as

$$q_{2,1}(L) = L^{d - n(d-2)/2 - p}$$

$$= L^{3 - 2(3-2)/2 - 1}$$

$$= L^{3 - 1 - 1}$$

$$= L q_{2,1} = L q_{2,1}$$

so it gets stronger as $L \to \infty$.

The Maxwell terms is $g_{2,2}$ and

$$q_{2,2}(L) = L^{3 - 2(3-2)/2 - 2}$$

$$g_{2,2} = q_{2,2}.$$
5. the Maxwell term is less important as $L \to 0$

than the Chern-Simons term.

We now add to the C-S gauge field an
an electromagnetic field $A_m$ so $L$ is

$$L = \frac{\hbar}{4\pi} \varepsilon_{\mu \nu \lambda} \text{and} d\alpha + \frac{i}{2\pi} \varepsilon_{\mu \nu \lambda} \text{and} d\alpha$$

$$= \frac{\hbar}{4\pi} \varepsilon_{\mu \nu \lambda} \text{and} d\alpha - \frac{i}{2\pi} \varepsilon_{\mu \nu \lambda} \text{and} d\alpha.$$  

$A_\lambda$ is not the one whose curl is $\tilde{B}$; that's
already in $h$. Note the integration by parts

$$\varepsilon_{\mu \nu \lambda} \text{and} d\alpha \rightarrow - \varepsilon_{\mu \nu \lambda} \text{and} d\alpha = - \varepsilon_{\mu \nu \lambda} \text{and} d\alpha$$

$$= - \varepsilon_{\mu \nu \lambda} \text{and} d\alpha = \varepsilon_{\mu \nu \lambda} \text{and} d\alpha.$$

Adding in the current $j^\mu$ of quasi-particles

$$L = \frac{\hbar}{4\pi} \varepsilon_{\mu \nu \lambda} \text{and} d\alpha \text{and} j^\mu + \frac{i}{2\pi} \varepsilon_{\mu \nu \lambda} \text{and} d\alpha.$$

Define by $j^\mu$ what couples to $a_\mu$:

$$L = \frac{\hbar}{4\pi} \varepsilon_{\mu \nu \lambda} \text{and} d\alpha + a_\mu j^\mu \text{with}$$

$$j^\mu = j^\mu - \frac{i}{2\pi} \varepsilon_{\mu \nu \lambda} \text{and} d\alpha.$$
If we integrate out the C-S a^m, we get

\[ \mathcal{L} = \frac{\pi}{\hbar} \int \text{d}^3 \phi \left( \frac{1}{2} \phi^2 - \frac{1}{2} \frac{\nabla \phi \cdot \nabla \phi}{\rho^2} - \frac{1}{2} \frac{\nabla A \cdot \nabla A}{\rho^2} \right) \]

Here, as before, we're using

\[ \int \mathcal{D} \phi \ e^{-i \int \mathcal{L}} = e^{i \int \mathcal{L}} \]

So \( \mathcal{L} \) now is

\[ \mathcal{L} = \frac{\pi}{\hbar} \left( j^m - \frac{i}{2 \pi} \varepsilon^{mn \lambda} \partial_n A_\lambda \right) \frac{\varepsilon_{\mu \nu 0}}{\pi} \left( j^\lambda - \frac{i}{2 \pi} \varepsilon_{\mu \nu 0} \partial_\nu A_\mu \right) \]

We have \( j^\mu, J^A, \) and \( A^A \) terms here.

The \( AA \) part is, since \( \varepsilon \varepsilon \varepsilon = \delta^2 \)

\[ A \left( \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon / \delta^2 \right) A = A \varepsilon \varepsilon \delta A \]

So

\[ \mathcal{L}_A = -\frac{1}{4 \pi \hbar} \varepsilon^{\mu \nu \lambda} \partial_\mu A_\nu \partial_\lambda A_\lambda \]

and so the electromagnetic current is

\[ j^\mu = -\frac{1}{4 \pi \hbar} \varepsilon^{\mu \nu \lambda} \partial_\nu A_\lambda , \]
\[ S_0 = \frac{\epsilon}{\hbar} \cdot \partial \cdot A \]

\[ = \frac{1}{4\pi \hbar} \left( \partial_1 A_2 - \partial_2 A_1 \right) = \frac{B_2}{4\pi \hbar} \]

A fluctuating SB implies an excess of

\[ S_n = \frac{1}{2\pi \hbar} \text{SB electrons.} \]

So the filling factor is \( \nu = \frac{1}{2} \).

Next,

\[ j_{\text{em}}^i = \frac{\epsilon}{\hbar} \cdot \partial \cdot A \]

\[ = \frac{1}{4\pi \hbar} \left( \epsilon \epsilon' \right) \partial_1 A_2 = \frac{1}{4\pi \hbar} \left( e_2 A_{02} - e_0 A_2 \right) \]

\[ = \frac{E_2}{4\pi \hbar} \quad \text{So} \quad \nu_{12} = \frac{1}{\frac{1}{2}} = \nu. \]

The Aj term is \( A_c (\epsilon e d e d / d^2) j \), so

\[ L_{\text{Aj}} = -\frac{1}{\hbar} \frac{A_{\text{Aj}}}{\hbar} \]

so the quasiparticle \( j^\prime \) carries charge \( q = \frac{1}{\nu} \).
Finally, the \( j^n \) of the quasi-particles in action

\[
\hat{\mathcal{L}}_{jj} = \frac{\pi}{\hbar} \int \sum_{\nu} v_f^2 j^\nu \frac{d^2}{d^2}
\]

So the quasi-particles obey "fractional" statistics with

\[
\frac{\Theta}{\pi} = \frac{1}{h} = \nu.
\]

But why should \( \nu \) be an odd integer?

The action

\[\hat{\mathcal{L}}_{jj} = \frac{1}{\hbar} A \cdot \hat{\mathcal{L}}_j^m\]

tells us that a composite of \( h \) quasi-particles has charge 1. Is this the hole? (the hole).

If so, \( h \) must be an integer.

If one quasi-particle moves by \( \pi \) about another, the phase change is \( \frac{\Theta}{\pi} = \frac{1}{h} \).

But if we move a composite of \( h \) quasi-particles about another composite of \( h \) quasi-particles, then

\[
\frac{\Theta}{\pi} = \frac{1}{\hbar} h^2 = h \quad \text{or} \quad \frac{\Theta}{\pi} = h.
\]
So if the composite of $n$ $p$'s is to be a fermion, we need $\frac{n}{2} = 2m + 1$.

So $h = 2m + 1 = \frac{1}{v} \quad \Delta B$

If electrons in plane

have $v = \frac{1}{3}$, then each $e$ is like 3 pieces of charge $\frac{1}{3}$ and fractional statistics $\frac{1}{3}$.

People call this "topological order" because the C-S theory depends on the topology, not the metric, of the manifold.

The short-distance properties of a Hall fluid can depend upon other terms such as $F^2$ etc. and upon impurities.

An effective field theory like that of K-H-Z in eq. (4) says the quasiparticle is a vortex with elections whirling around it. This fits Wilczek's models.

Recall we dropped a surface term when we showed that the C-S action was gauge invariant. But real Hall fluids do have quite finite edge.

There are physical degrees of freedom on the edges whose action cancels the surface term we dropped.

An incompressible fluid has edge excitations like waves on its boundary.

The current $j^a$ has dimension $L^{-2}$ so we make $g j^a$ have dimension $L^{-3}$ or $m^3$. So $j a j^a \sim (\text{mass})^2$ which is the same as the Maxwell term $F^2$.

We can't make $L$ of $m^3$ out of $j^a$ alone. That's why we need $J(1/6)J$, which is the Hopf term which is nonlocal.

We use gauge fields to make the theory local.

Double-layered Hall systems allow tunneling, $J_i^a = \frac{1}{2} \epsilon^{abc} J_a J^b$. 
Now \[ J = \sum_{i,j} K_{ij} \phi_i^* \phi_j + \ldots \]

If \( K \) has a zero eigenvalue, then the Maxwell term \( F^2 \) becomes important and a superfluid appears, seen in experiments.

Some use a gas of monopoles and \( C-S \) antimonopoles to describe the tunneling of the electrons from layer \( F \) to layer \( J \).