Chapter 11

Functionals

11.1 Functionals

A functional $G[f]$ is a map from a space of functions to a set of numbers. For instance, the action functional $S[q]$ for a particle in one dimension maps the coordinate $q(t)$, which is a function of the time $t$, into a number—the action of the process. If the particle of mass $m$ is moving slowly and freely, then for the interval $(t_1, t_2)$ its action is

$$S[q] = \int_{t_1}^{t_2} dt \; \frac{m}{2} \left( \frac{dq(t)}{dt} \right)^2.$$  \hspace{1cm} (11.1)

If the particle is moving in a potential $V(q(t))$, then its action is

$$S[q] = \int_{t_1}^{t_2} dt \left[ \frac{m}{2} \left( \frac{dq(t)}{dt} \right)^2 - V(q(t)) \right].$$  \hspace{1cm} (11.2)

11.2 Functional Derivatives

A functional derivative is a functional

$$\delta G[f][h] = \frac{d}{d\epsilon} G[f + \epsilon h] \bigg|_{\epsilon=0}$$  \hspace{1cm} (11.3)

of a functional. For instance, if $G[f]$ is the functional

$$G[f] = \int dx \, f''(x)$$  \hspace{1cm} (11.4)

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then its functional derivative is the functional that maps the pair of functions $f, h$ to the number

$$\frac{\delta G[f]}{\delta f(y)} = \delta G[f][\delta(x - y)]$$

in which the second function $h(x)$ is $\delta(x - y)$. Thus, in the preceding example

$$\frac{\delta G[f]}{\delta f(y)} = \int dx \, nf^{n-1}(x)\delta(x - y) = nf^{n-1}(y).$$

Functional derivatives of functionals that involve powers of derivatives also are easily dealt with. Suppose that the functional involves the square of the derivative $f'(x)$

$$G[f] = \int dx \, (f'(x))^2.$$ 

Then its functional derivative is

$$\frac{\delta G[f]}{\delta f(y)} = -2 \int dx \, f''(x)\delta(x - y) = -2f''(y).$$
11.2. FUNCTIONAL DERIVATIVES

Let's now compute the functional derivative of the action (11.2), which involves the square of the time-derivative \( \dot{q}(t) \) and the potential energy \( V(q(t)) \)

\[
\delta S[q|h] = \left. \frac{d}{d\epsilon} S[q + \epsilon h] \right|_{\epsilon=0} = \frac{d}{d\epsilon} \int dt \left[ \frac{m}{2} \left( \dot{q}(t) + \epsilon \dot{q}(t) \right)^2 - V(q(t) + \epsilon h(t)) \right]_{\epsilon=0} = \int dt \left[ m\ddot{q}(t)\dot{h}(t) - V'(q(t))h(t) \right] = \int dt \left[ -m\dddot{q}(t) - V'(q(t)) \right] h(t)
\]

(11.11)

where we once again have integrated by parts and used suitable boundary conditions to drop the surface terms. In physics notation, this is

\[
\frac{\delta S[q]}{\delta q(t')} = \int dt \left[ -m\dddot{q}(t) - V'(q(t)) \right] \delta(t - t') = -m\dddot{q}(t') - V'(q(t')).
\]

(11.12)

In these terms, the stationarity of the action \( S[q] \) is the vanishing of its functional derivative

\[
\frac{\delta S[q]}{\delta q(t)} = 0
\]

(11.13)

which is Lagrange’s equation of motion

\[
m\dddot{q} = -V'(q).
\]

(11.14)

Here's a shortcut to the functional derivative in the notation of physics

\[
\frac{\delta G[f]}{\delta f(y)} = \left. \frac{d}{d\epsilon} G[f + \epsilon \delta_y] \right|_{\epsilon=0}
\]

(11.15)

in which the function \( h(x) \) has been replaced by \( \delta_y(x) = \delta(x - y) \).