

# Supersymmetric Quantum Mechanics

First, let's go one step beyond the simple harmonic oscillator

$$H_0 = \hbar\omega \left( a^\dagger a + \frac{1}{2} \right) \quad \text{where}$$

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left( x + \frac{i p}{m\omega} \right).$$

We may replace  $x$  by  $f(x)$  so that

$$H = \hbar\omega \left( b^\dagger b + c \right) \quad \text{where now}$$

$$b = \sqrt{\frac{m\omega}{2\hbar}} \left( f(x) + \frac{i p}{m\omega} \right).$$

Now

$$H = \hbar\omega \left[ \left( \frac{m\omega}{2\hbar} \right) \left( f(x) - \frac{i p}{m\omega} \right) \left( f(x) + \frac{i p}{m\omega} \right) + c \right]$$

$$= \hbar\omega \left[ \left( \frac{m\omega}{2\hbar} \right) \left( f^2(x) + \frac{p^2}{m^2\omega^2} + \frac{i}{m\omega} [f(x), p] \right) + c \right]$$

$$= \hbar\omega \left[ \left( \frac{m\omega}{2\hbar} \right) \left( f^2(x) + \frac{p^2}{m^2\omega^2} - \frac{\hbar}{m\omega} f'(x) \right) + c \right]$$

$$= \frac{p^2}{2m} + \frac{1}{2} m\omega^2 f^2(x) - \frac{\hbar\omega}{2} f'(x) + \hbar\omega c.$$

Once again, the ground state  $|0\rangle$  satisfies

$$\langle x | b | 0 \rangle = \sqrt{\frac{m\omega}{2\hbar}} \langle x | f(x) + \frac{i\hbar}{m\omega} \frac{d}{dx} | 0 \rangle = 0$$

or

$$f(x) \langle x | 0 \rangle = -\frac{i\hbar}{m\omega} \frac{d}{dx} \langle x | 0 \rangle$$

(on with prime meaning  $d/dx$ )

$$\frac{\langle x | 0 \rangle'}{\langle x | 0 \rangle} = -\frac{m\omega}{\hbar} f(x)$$

$$\log \langle x | 0 \rangle = -\frac{m\omega}{\hbar} \int_0^x dx' f(x') + C$$

$$\langle x | 0 \rangle = N \exp \left[ -\frac{m\omega}{\hbar} \int_0^x dx' f(x') \right]$$

This solution makes sense if

$f(x) \rightarrow +\infty$  as  $x \rightarrow \infty$  and

$f(x) \rightarrow -\infty$  as  $x \rightarrow -\infty$ . For example, if

$f(x) = dx^3$ , with  $d > 0$ , then

$$d \int_0^x dx' x'^3 = d \frac{x^4}{4} \quad \text{and so}$$

$$\langle x | 0 \rangle = N \exp \left( -\frac{m\omega d}{4\hbar} x^4 \right)$$

is the normalizable ground state with energy

$$H|0\rangle = \hbar\omega (b^\dagger b + c)|0\rangle \\ = \hbar\omega c|0\rangle.$$

So the ground-state energy depends upon the value of  $c$ .

But if  $f(x) = dx^3$  with  $d < 0$  or if  $f(x)$  is an even function of  $x$ , then  $b|0\rangle = 0$  will have no normalizable solution, and the ground state must be found by solving the second-order differential equation  $\langle x|H|0\rangle = E_0 \langle x|0\rangle$ .

Now supersymmetric (susy) quantum mechanics as invented by Edward Witten. Think of a spin- $1/2$  particle of mass  $m$  on a line. Consider the "charges"

$$Q_1 = \frac{1}{2} \left( \frac{\sigma_1 P}{\sqrt{m}} + \sigma_2 \sqrt{m} W(x) \right)$$

$$Q_2 = \frac{1}{2} \left( \frac{\sigma_2 P}{\sqrt{m}} - \sigma_1 \sqrt{m} W(x) \right).$$

Note that both are Hermitian

$$Q_1^\dagger = Q_1 \quad \text{and} \quad Q_2^\dagger = Q_2.$$

Algebra plays a big role in susy.

Note that

$$Q_1^2 = \frac{1}{4} \frac{\sigma_1^2 p^2}{m} + \frac{1}{4} \sigma_2^2 m W^2(x) + \frac{1}{2} \sigma_1 p \sigma_2 W(x) + \frac{1}{2} \sigma_2 W(x) \sigma_1 p$$

Now

$$\sigma_i \sigma_j = \delta_{ij} + \sum_{k=1}^3 i \epsilon_{ijk} \sigma_k$$

So

$$\sigma_1 \sigma_2 = i \sigma_3 \quad \text{and} \quad \sigma_2 \sigma_1 = -i \sigma_3$$

while

$$\sigma_1^2 = \sigma_2^2 = 1.$$

So

$$Q_1^2 = \frac{1}{4} \frac{p^2}{m} + \frac{1}{4} m W^2(x) + \frac{i \sigma_3}{2} p W(x) - \frac{i \sigma_3}{2} W(x) p$$

$$= \frac{1}{4} \frac{p^2}{m} + \frac{1}{4} m W^2(x) - \frac{i \sigma_3}{2} [W(x), p]$$

$$= \frac{1}{4} \frac{p^2}{m} + \frac{1}{4} m W^2(x) + \frac{\hbar \sigma_3}{2} W'(x)$$

where  $W'(x) = dW(x)/dx$ . Similarly

$$Q_2^2 = \frac{1}{4} \frac{p^2}{m} + \frac{1}{4} m W^2(x) - \frac{1}{2} \sigma_2 p \sigma_1 W(x) - \frac{1}{2} \sigma_1 W(x) \sigma_2 p$$

$$= \frac{1}{4} \frac{p^2}{m} + \frac{1}{4} m W^2(x) + \frac{1}{2} i \sigma_3 [W(x), p]$$

$$= \frac{1}{4} \frac{p^2}{m} + \frac{1}{4} m W^2(x) + \frac{\hbar}{2} \sigma_3 W'(x) = Q_1^2$$

The anti commutator  $\{Q_1, Q_2\} = 0$ .

$$Q_1 Q_2 = \frac{1}{4} \left( \frac{\sigma_1 p}{\sqrt{m}} + \sigma_2 \sqrt{m} W(x) \right) \left( \frac{\sigma_2 p}{\sqrt{m}} - \sigma_1 \sqrt{m} W(x) \right)$$

$$= \frac{1}{4} \frac{p^2}{m} \sigma_1 \sigma_2 - m W(x)^2 \sigma_2 \sigma_1 + W(x) p - p W(x)$$

$$Q_2 Q_1 = \frac{1}{4} \frac{p^2}{m} \sigma_2 \sigma_1 - m W(x)^2 \sigma_1 \sigma_2 - W(x) p + p W(x)$$

So  $\sigma_i \sigma_j = \delta_{ij} + \sum_k i \epsilon_{ijk} \sigma_k \Rightarrow \sigma_1 \sigma_2 + \sigma_2 \sigma_1 = 0$

Thus

$$Q_1 Q_2 + Q_2 Q_1 = 0$$

The susy algebra here is

$$[Q_i, Q_j]_+ = \delta_{ij} H = \{Q_i, Q_j\}.$$

and

$$[Q_i, H] = 0, \quad S_0$$

$$H = 2Q_1^2 = 2Q_2^2$$

$$= \frac{p^2}{2m} + \frac{m}{2} W^2(x) - \frac{\hbar}{2} \sigma_3 W'(x).$$

We say "susy is exact" if  $Q_1$  or  $Q_2$  annihilates the ground state  $|0\rangle$ .

$$Q_1 |0\rangle = 0 \quad \text{or} \quad Q_2 |0\rangle = 0.$$

In both cases

$$H |0\rangle = 2Q_1^2 |0\rangle = 0 = 2Q_2^2 |0\rangle.$$

And so both

$$\langle 0 | Q_1^2 |0\rangle = 0 = \langle 0 | Q_2^2 |0\rangle,$$

which means both

$$Q_1 |0\rangle = 0 \quad \text{and} \quad Q_2 |0\rangle = 0.$$

Let's solve

$$0 = \langle x | Q_1 |0\rangle = \frac{1}{2} \left( \frac{\sigma_1}{\sqrt{m}} \langle x | p |0\rangle + \sigma_2 \sqrt{m} W(x) \langle x |0\rangle \right)$$

$$\frac{\hbar}{i} \frac{\sigma_1}{\sqrt{m}} \langle x |0\rangle' + \sigma_2 \sqrt{m} W(x) \langle x |0\rangle = 0 \quad \text{or}$$

$$\frac{\hbar}{i} \langle x |0\rangle' = -i \sigma_3 m W(x) \langle x |0\rangle$$

in which  $\langle x |0\rangle$  has two components,

So  $\langle x|0\rangle' = \frac{m}{\hbar} \sigma_3 W(x) \langle x|0\rangle$

which we integrate (as before) to

$$\langle x|0\rangle = N \exp \left[ \int_0^x dx' \frac{m}{\hbar} \sigma_3 W(x') \right] \langle 0|0\rangle.$$

As for the case in which  $H = \hbar\omega(b^\dagger b + c)$ , we can find a solution if the leading power of  $W(x)$  is odd — but not if it is even. For instance, if

$$W(x) = c_3 x^3 + c_2 x^2 + c_1 x + c_0$$

then

$$\int_0^x dx' W(x') = \frac{c_3}{4} x^4 + \frac{c_2}{3} x^3 + \frac{c_1}{2} x^2 + c_0 x$$

and

$$\langle x|0\rangle = N \exp \left[ \frac{m}{\hbar} \sigma_3 \left( \frac{c_3}{4} x^4 + \frac{c_2}{3} x^3 + \frac{c_1}{2} x^2 + c_0 x \right) \right] \times \langle 0|0\rangle.$$

This may look bad if  $c_3 > 0$ , but recall that  $\langle 0|0\rangle$  is a spinor. We choose if  $c_3 > 0$

$$\langle 0|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ in which case}$$

$$\langle x|0\rangle = N \exp \left[ -\frac{m}{\hbar} \left( \frac{c_3}{4} x^4 + \frac{c_2}{3} x^3 + \frac{c_1}{2} x^2 + c_0 x \right) \right] \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

So if  $W(x) = c_4 x^4$ , for instance, then

$$Q_1 |0\rangle = 0$$

has no solution, and we must instead solve

$$H|0\rangle = 2Q_1^2 |0\rangle = E_0 |0\rangle$$

which is much harder to do. Also, the energy  $E_0$  must be positive, since  $E_0 > 0$  and  $E_0 = 0$  would imply

$$\langle 0 | Q_1^2 |0\rangle = \frac{E_0}{2} \langle 0 |0\rangle = 0$$

which implies

$$Q_1 |0\rangle = 0,$$

which we saw was impossible when the leading power of  $W(x)$  is even.

$Q_1$  and  $Q_2$  generate susy transformations. When

$$Q_1 |0\rangle = Q_2 |0\rangle = 0,$$

the ground state is invariant under susy transformations and we say susy is exact. In this case,

$$\begin{aligned} & \langle 0 | e^{-\alpha Q_1 - \beta Q_2} A B \dots Z e^{\alpha Q_1 + \beta Q_2} |0\rangle \\ &= \langle 0 | e^{\alpha Q_1 + \beta Q_2} A B \dots Z e^{-\alpha Q_1 - \beta Q_2} |0\rangle \\ &= \langle 0 | A B \dots Z |0\rangle. \end{aligned}$$