Rotations

Under a right-handed rotation by ~~a~~ small angle θ about the axis $\vec{\theta}$ the vector $\vec{r}$ changes to

$$\vec{r} + \vec{\delta r} = \vec{r} + \vec{\theta} \times \vec{r}.$$

Here $\vec{\theta} \cdot \vec{\theta} = \theta^2$ and

$$(\theta \times r)_k = \sum_{i,j=1}^{3} \epsilon_{kij}\, \theta_i\, r_j$$

where ϵ_{ijk} is Levi-Civita's tensor which is totally anti-symmetric with $\epsilon_{123} = 1$.

So

$$\langle \vec{r} + \vec{\delta r} | = \langle \vec{r} | \, e^{i \vec{\delta r} \cdot \vec{p}/\hbar}$$

$$\shortparallel$$

$$\langle \vec{r} + \vec{\theta} \times \vec{r} | = \langle \vec{r} | \, e^{i (\overrightarrow{\theta \times r}) \cdot \vec{p}/\hbar}.$$

Because ϵ_{ijk} is unchanged by a cyclic permutation

$$(\overrightarrow{\theta \times r}) \cdot \vec{p} = \sum_{i,j,k=1}^{3} \epsilon_{kij}\, \theta_i\, r_j\, p_k$$

$$= \sum_{i,j,k=1}^{3} \epsilon_{ijk}\, \theta_i\, r_j\, p_k = \vec{\theta} \cdot (\vec{r} \times \vec{p}).$$

So

$$\langle \vec{r} + \vec{\theta} \times \vec{r} \,| = \langle \vec{r}\,| \, e^{i \vec{\theta} \cdot \vec{L} / \hbar}$$

where the angular momentum vector $\vec{L}$ is

$$L_i = \sum_{j,k=1}^{3} \epsilon_{ijk} \, r_j \, p_k$$

or

$$\vec{L} = \vec{r} \times \vec{p} .$$

The adjoint of the top equation is

$$| r + \theta \times r \rangle = e^{-i \vec{\theta} \cdot \vec{L} / \hbar} \, |\vec{r}\rangle ,$$

so the unitary operator that rotates states by $\theta = \|\vec{\theta}\| = \sqrt{\vec{\theta} \cdot \vec{\theta}}$ in the right-handed sense is

$$U(\vec{\theta}) = e^{-i \vec{\theta} \cdot \vec{L} / \hbar} .$$

This angular-momentum operator $\vec{L}$ pertains to orbital angular momentum. If the particle has spin, then the full angular momentum operator is

$\vec{J} = \vec{L} + \vec{S}$ where $\vec{S} = \frac{\hbar}{2} \vec{\sigma}$ for spin $\frac{\hbar}{2}$.

But the <u>real</u> definition of angular momentum $\vec{J}$ is the operator that generates rotations. In gauge theories, the gauge fields contribute to $\vec{J}$.

Demo: In class, I will rotate a big, black sphere about the x-axis and then about the y-axis. Then I will apply the inverse rotations about the x and y axes. If rotations in three dimensions commuted, the sphere would return to its original position. It does not.

In symbols: For tiny $\epsilon = \theta/\hbar$ and $\delta = \theta/\hbar$

$$e^{i\delta L_y}\, e^{i\epsilon L_x}\, e^{-i\delta L_y}\, e^{-i\epsilon L_x}$$

$$= \left(1 + i\delta L_y - \frac{\delta^2}{2} L_y^2\right)\left(1 + i\epsilon L_x - \frac{\epsilon^2}{2} L_x^2\right)$$

$$\left(1 - i\delta L_y - \frac{\delta^2}{2} L_y^2\right)\left(1 - i\epsilon L_x - \frac{\epsilon^2}{2} L_x^2\right)$$

plus higher-order terms in ϵ & δ

$$= \left(1 + i\delta L_y + i\epsilon L_x - \delta\epsilon L_y L_x - \frac{\delta^2}{2} L_y^2 - \frac{\epsilon^2}{2} L_x^2\right)$$

$$\left(1 - i\delta L_y - i\epsilon L_x - \delta\epsilon L_y L_x - \frac{\delta^2}{2} L_y^2 - \frac{\epsilon^2}{2} L_x^2\right)$$

$$= 1 - 2\delta\epsilon L_y L_x + \delta\epsilon L_y L_x + \delta\epsilon L_x L_y$$
$$+ \delta^2 L_y^2 + \epsilon^2 L_x^2 - \delta^2 L_y^2 - \epsilon^2 L_x^2$$

Thus for tiny rotations

$$e^{i\delta L_y} e^{i\epsilon L_x} e^{-i\delta L_y} e^{-i\epsilon L_x} = 1 + \delta\epsilon [L_x, L_y].$$

The demo shows that the net result of these four rotations is a tiny left-handed rotation about the z-axis:

$$e^{i\delta L_y} e^{i\epsilon L_x} e^{-i\delta L_y} e^{-i\epsilon L_x} = e^{+i\hbar\delta\epsilon L_z}$$

$$= 1 + i\hbar\delta\epsilon L_z.$$

So we expect that

$$[L_x, L_y] = i\hbar L_z$$

for in this case

$$\delta\epsilon [L_x, L_y] = i\hbar\delta\epsilon L_z.$$

Going back to θ and θ', we have

$$\frac{\theta\theta'}{\hbar^2} [L_x, L_y] = i\hbar \frac{\theta\theta'}{\hbar^2} L_z$$

or

$$[L_x, L_y] = i\hbar L_z.$$

We may verify this commutation relation by noting that since

$$\vec{L} = \vec{r} \times \vec{p}$$

$$L_x = L_1 = \sum_{ij=1}^{3} \epsilon_{1ij} r_i p_j = r_2 p_3 - r_3 p_2$$

and

$$L_y = L_2 = \sum_{ij=1}^{3} \epsilon_{2ij} r_i p_j = \epsilon_{213} r_1 p_3 + \epsilon_{231} r_3 p_1$$

$$= -r_1 p_3 + r_3 p_1 .$$

Since $[r_j, p_k] = i\hbar \delta_{ij}$, one has

$$[L_x, L_y] = [r_2 p_3 - r_3 p_2, -r_1 p_3 + r_3 p_1]$$

$$= r_2 p_1 [p_3, r_3] + p_2 r_1 [r_3, p_3]$$

$$= -i\hbar r_2 p_1 + i\hbar p_2 r_1 = i\hbar \left(\epsilon_{312} r_1 p_2 + \epsilon_{321} r_2 p_1 \right)$$

$$= i\hbar \sum_{jk=1}^{3} \epsilon_{3jk} r_j p_k = i\hbar L_3 = i\hbar L_z .$$

This is an instance of the general formula

$$[L_i, L_j] = i\hbar \sum_{k=1}^{3} \epsilon_{ijk} L_k .$$

This last formula is satisfied by the spin-1/2 angular-momentum operators

$$S_i = \frac{\hbar}{2}\sigma_i .$$

The rule $\quad \sigma_i \sigma_j = \delta_{ij} + i \sum_{k=1}^{3} \epsilon_{ijk}\sigma_k$

implies that

$$[S_i, S_j] = \frac{\hbar^2}{4}\left(\delta_{ij} + i\sum \epsilon_{ijk}\sigma_k - \delta_{ij} - \sum \epsilon_{jik}\sigma_k\right)$$

$$= \frac{i\hbar^2}{2}\sum \epsilon_{ijk}\sigma_k$$

$$= i\hbar \sum_{k=1}^{3} \epsilon_{ijk} S_k . \qquad (1.4.20)$$

More generally,

$$[J_i, J_j] = i\hbar \sum_{k=1}^{3} \epsilon_{ijk} J_k .$$

In fact any operator that transforms as a vector $\vec{V}$ under rotations

$$\vec{V} \to \vec{V} + \vec{\theta} \times \vec{V}$$

will satisfy

$$[J_i, V_j] = i\hbar \sum_{k=1}^{3} \epsilon_{ijk} V_k.$$

Since $\vec{V}$ behaves as a vector under a tiny rotation by $\vec{\theta}$, we have

$$e^{\frac{i\theta\cdot J}{\hbar}} V_j e^{-i\theta\cdot J/\hbar} = V_j + (\theta \times V)_j$$

$$\shortparallel$$

$$V_j + \frac{i\theta_i}{\hbar}\sum_{i=1}^{3} [J_i, V_j] = V_j - \sum_{i,l=1}^{3} \epsilon_{ijk}\, \theta_i V_k$$

or

$$\frac{i}{\hbar}[J_i, V_j] = -\sum_{k=1}^{3} \epsilon_{ijk} V_k$$

$$[J_i, V_j] = i\hbar \sum_{k=1}^{3} \epsilon_{ijk} V_k.$$

So the rule

$$[J_i, V_j] = i\hbar \sum_{k=1}^{3} \epsilon_{ijk} V_k$$

says that $\vec{V}$ is a vector under rotations.

Scalars are invariant under rotations; if S is a scalar then

$$[J_i, S] = 0.$$

Suppose $\vec{u}$ and $\vec{V}$ are two vectors

$$[J_i, V_j] = i\hbar \sum_k \epsilon_{ijk} V_k$$

$$[J_i, u_j] = i\hbar \sum_k \epsilon_{ijk} U_k.$$

Then their dot product $\vec{u}\cdot\vec{V}$ is a scalar:

$$[J_i, \sum_j u_j V_j] = \sum_j \left(J_i u_j V_j - u_j J_i V_j + u_j J_i V_j - u_j V_j J_i\right)$$

$$= \sum_j [J_i, u_j] V_j + u_j [J_i, V_j]$$

$$= i\hbar \sum_{jk=1}^{3} \epsilon_{ijk} u_k V_j + \epsilon_{ijk} u_j V_k$$

$$= i\hbar \sum_{jk} \epsilon_{ikj} u_j V_k + \epsilon_{ijk} u_j V_k$$

$$= i\hbar \sum_{jk} \left(\epsilon_{ikj} + \epsilon_{ijk}\right) u_j V_k = 0.$$

Thus $\sum_j J_j^2 = \vec{J}\cdot\vec{J}$ is a scalar, and

$$[J_i, \vec{J}^2] = 0.$$