A Relativistic Particle in an Electromagnetic Field

The lagrangian for a particle of mass m and charge q in an electromagnetic field $A_b = (\vec{A}, A_0)$ is

$$L = -mc\sqrt{\frac{dx_a}{dt}\frac{dx^a}{dt}} + q A_b \frac{dx^b}{dt}$$

so the action S in SI units is

$$S = \int L dt = -mc\int\sqrt{dx_a dx^a} + q\int A_b dx^b .$$

Here

$$\begin{aligned} dx_a dx^a &= c^2 dt^2 - d\vec{x}^{\,2} \\ &= c^2 dt^2 (1 - \vec{v}^{\,2}/c^2) \end{aligned}$$

in which

$$\vec{v} = \frac{d\vec{x}}{dt}$$

is the velocity of the particle. The field A^0 is the electric potential ϕ divided by c

$$A^0 = \frac{\phi}{c} \quad \text{so that} \quad A_0 = -\frac{\phi}{c} .$$

So the action is

$$S = \int dt \left[-mc^2\sqrt{1 - \frac{\vec{v}^{\,2}}{c^2}} + q\vec{A}\cdot\vec{v} - q\phi \right] .$$

Thus another way to write L is

$$L = -mc^2\sqrt{1-\frac{v^2}{c^2}} + q\vec{A}\cdot\vec{v} - q\phi .$$

Now the momentum canonically conjugate to $\vec{x}$ is

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} + q\vec{A} .$$

So the mechanical momentum is

$$\vec{P}_{mech} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} = \vec{p} - q\vec{A} .$$

To find the hamiltonian H, we must express $\vec{v}$ in terms of $\vec{p}$. We have

$$\frac{\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{\vec{p}-q\vec{A}}{m} \quad \text{or} \quad \frac{v^2}{1-\frac{v^2}{c^2}} = \frac{(\vec{p}-q\vec{A})^2}{m^2} .$$

So

$$v^2 = \frac{(p-qA)^2/m^2}{1+(p-qA)^2/(m^2c^2)}$$ whence

$$1-\frac{v^2}{c^2} = \frac{1}{1+\frac{(p-qA)^2}{m^2c^2}} .$$

So

$$\sqrt{1-\frac{v^2}{c^2}} = \frac{1}{\sqrt{1+\frac{(p-qA)^2}{m^2c^2}}}$$ and so

$$\vec{v} = \frac{1}{m}(\vec{p}-q\vec{A})\sqrt{1-\frac{v^2}{c^2}}$$

$$= \frac{1}{m}\frac{(\vec{p}-q\vec{A})}{\sqrt{1+\frac{(p-qA)^2}{m^2c^2}}}.$$

Finally then

$$H = p\cdot v - L = \frac{p\cdot(p-qA)}{m\sqrt{1+\frac{(p-qA)^2}{m^2c^2}}} + mc^2\frac{1}{\sqrt{1+\frac{(p-qA)^2}{m^2c^2}}}$$

$$-\frac{q\vec{A}\cdot(\vec{p}-q\vec{A})}{m\sqrt{1+\frac{(p-qA)^2}{m^2c^2}}} + q\phi$$

$$= \frac{\left[mc^2+\frac{1}{m}(\vec{p}-q\vec{A})^2\right]}{\sqrt{1+\frac{(\vec{p}-q\vec{A})^2}{m^2c^2}}} + q\phi$$

or

$$H = \frac{m^2c^2 + (\vec{p} - q\vec{A})^2}{\frac{1}{c}\sqrt{m^2c^2 + (\vec{p} - q\vec{A})^2}} + q\phi$$

$$= c\sqrt{m^2c^2 + (\vec{p} - q\vec{A})^2} + q\phi.$$

More explicitly,

$$H = c\sqrt{m^2c^2 + \left(\vec{p} - q\vec{A}(x,t)\right)^2} + q\phi(\vec{x},t).$$

The non-relativistic form of the square-root gives us

$$H = mc^2 + \frac{(\vec{p} - q\vec{A})^2}{2m} + q\phi.$$

If we add spin, and drop mc^2, we get

$$H = \frac{(\vec{p} - q\vec{A})^2}{2m} + q\phi - \frac{q}{m}\vec{S}\cdot\vec{B}$$

in which $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$ and $B = \nabla \times A$.