The Quadratic Stark Effect on the Ground State of H

\[ H_0 = \frac{p^2}{2\mu} - \frac{e^2}{r} \]

\[ \Delta V = e^2 \frac{z^2}{\lambda} \quad (\ell > 0) \]

The electric field is parallel to the z-axis.

The mean value of \( z \) vanishes in the spherically symmetric ground state:

\[ \langle 1001z^21100 \rangle = 0 \]

so to first order in \( e^2 \) the energy of the ground state does not change.

\[ \Delta E_1 = e^2 \frac{z^2}{\lambda} \langle 1001z1100 \rangle = 0. \]

To second order in \( e^2 \), the change in the energy is

\[ \Delta E_{100}^2 = e^2 \frac{z^2}{\lambda} \sum_{\ell m} |\langle n\ell m|z1100 \rangle| \frac{1}{E_0 - E_n^0} < 0 \]

in which \( E_n^0 = -EI/n^2 = -\frac{\ell}{2} \mu c^2 \alpha^2/m^2 \).
What is the electric dipole moment of this H-atom? To lowest order in CE,
\[ \vec{d} = \langle 41 - e r^2 \rangle 14 \]

\[ \vec{d} = \langle \text{which since} \rangle \]

\[ |m_o\rangle + \lambda |m\rangle = |m_o\rangle + \frac{\phi_n}{E_{m} - H_0} \langle m_o | \lambda |m_o\rangle \]

\[ \langle \lambda \rangle = |1000\rangle + e \sum_{n+1 \leq \ell, m} \frac{|nlm\rangle \langle nm \lambda | z_{1100}\rangle}{E_{m} - E_{n}} \]

So
\[ \vec{d} = -e \langle 1001 \rangle \vec{1100} \]

\[ -e^2 \sum_{n+1 \leq \ell, m} \frac{\langle 1001 \rangle |nlm\rangle \langle nm | x_{nm} | z_{1100}\rangle + \langle 1001 \rangle |mlm\rangle \langle m \lambda | z_{1100}\rangle}{E_{m} - E_{n}} \]

Now
\[ \langle nlm | z_{1100}\rangle = 0 \text{ unless } m = 0 \text{ and } l = 1, \]
because
\[ z \propto \gamma^0_1, \]
Also, \( x \& y \) are proportional to linear
combinations of $Y_{l}^{m}$ so

$$<m|101x|1100> = 0 = <m|101y|1100>$$

so $d = d \epsilon$ with

$$d = -2e^{2} \sum_{n \geq 1} \frac{|<n|01z|1100>|^{2}}{E_{n}^{0} - E_{n}^{0}}$$

The linear electric susceptibility $\chi$ is given by

$$d = \chi \epsilon$$

so $\chi$ is

$$\chi = -2e^{2} \sum_{n \geq 1} \frac{|<n|01z|1100>|^{2}}{E_{n}^{0} - E_{n}^{0}}$$

Now

$$E_{n}^{0} - E_{n}^{0} = -\frac{1}{2} m c^{2} a^{2} (1 - \frac{1}{m^{2}})$$

so

$$\chi = 4e^{2} \sum_{n \geq 1} \frac{|<n|01z|1100>|^{2}}{1 - \frac{1}{m^{2}}}$$

$$= 4e^{2} \frac{c^{2}}{m c^{2} e^{4}} \sum_{n \geq 1} \frac{|<n|01z|1100>|^{2}}{1 - \frac{1}{m^{2}}}$$

$$= \frac{4 e^{2}}{m e^{2}} \sum_{n \geq 1} \frac{|<n|01z|1100>|^{2}}{1 - \frac{1}{m^{2}}}$$
Now for \( n > 1 \)

\[
\frac{1}{1 - \frac{1}{n^2}} = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3}
\]

so

\[
\chi_{15} \leq \frac{4}{\alpha^2} \sum_{n > 1} \langle 100 \mid z^{1/2} (n \mid 100) \rangle
\]

But the states \( 1n_{10} \) are the only ones that do not vanish in the above sum. Thus

\[
\chi_{15} \leq \frac{16}{3} \frac{t^2}{\alpha^2} \sum_{n \neq n_{10}} \langle 100 \mid z^{1/2} (n \mid 100) \rangle
\]

\[
\leq \frac{16}{3} \frac{t^2}{\alpha^2} \langle 100 \mid z^2 (1100) \rangle
\]

\[
\leq \frac{16}{3} a_0 \langle 100 \mid z^2 (1100) \rangle
\]

where \( a_0 = \frac{\hbar^2}{\alpha m \alpha^2} \) is the Bohr radius; \( a_0 \approx 0.53 \AA \).

Now

\[
\langle 100 \mid z^2 (1100) \rangle = \frac{1}{5} \langle 100 \mid \vec{v}^2 (1100) \rangle = \frac{1}{3} \int d^3 \vec{r} R_{10}(\vec{r}) \vec{r}^2
\]

\[
= \frac{4}{3} a_0^{-3} \int_0^\infty dr r^4 e^{-2r/a_0} = \left( \frac{a_0}{2} \right)^4 \frac{4}{3} a_0^{-3} \int_0^\infty dx x^4 e^{-x} = \frac{4}{3} \frac{4! a_0^2}{2^5}
\]

\[
= \frac{4 \cdot 4!}{3 \cdot 4!} a_0^2 = a_0^2
\]
So the linear electric susceptibility of the ground state of hydrogen is bounded by

$$\chi_{15} \leq \frac{16}{3} a_0^3 \approx 5.3 a_0^3,$$

The exact value is

$$\chi_{15} = 4.5 a_0^3.$$