1. Consider the scattering of a (spinless) particle off a square-well potential as in the class notes. Assume that $k_0 r_0$ is very close to an odd multiple of $\pi/2$. Set

$$k_0 r_0 = (2n+1)\frac{\pi}{2} + \epsilon \tag{1}$$

with $|\epsilon| \ll 1$. Show that if $\epsilon > 0$, then there is a bound state with energy

$$E = -\frac{(\hbar k'')^2}{2\mu} \tag{2}$$

with $k" \approx \epsilon k_0$.

The condition for a bound state is

$$\cot k' r_0 = -\frac{k''}{k'} \quad \text{where } k' = \sqrt{k_0^2 - k''^2}.$$

If $k_0 r_0 = (2n+1)\frac{\pi}{2} + \epsilon$, then to lowest order in $k''/k_0$

$\cot k' r_0 \simeq -\epsilon$ and so the bound-state condition is

$$-\epsilon = -\frac{k''}{k_0} \quad \text{to lowest order in } k''/k_0.$$

So the energy of the bound state is

$$E = -\frac{(\hbar k'')^2}{2\mu} = -\frac{(\epsilon \hbar k_0)^2}{2\mu}$$

with

$$k'' = \epsilon k_0.$$

2. Now in the previous problem change the sign of $\epsilon$, so that $\epsilon < 0$. Show that there is a bump in the s-wave total x-section (a scattering resonance) at energy

$$E = \frac{(\hbar k)^2}{2\mu} \tag{3}$$

with

$$k^2 \approx -\frac{2k_0\epsilon}{r_0}. \tag{4}$$

In effect, as the depth of the potential well is reduced, bound states become scattering resonances.

The s-wave phase shift $\delta_0(k)$ at momentum $k$ is related to the radius $r_0$ and the depth parameter $k_0$ by

$$k' \cot k' r_0 = k \cot(k r_0 + \delta_0(k)) \tag{1}$$

in which $k' = \sqrt{k_0^2 + k^2}$. The product $k_0 r_0$ is

$$k_0 r_0 = (2n+1)\frac{\pi}{2} + \epsilon \tag{2}$$

where $\epsilon < 0$. In these formulas, both $k$ and $\epsilon$ are infinitesimal, while $k_0$, $r_0$, and $\delta_0(k)$ are finite. For tiny $k$ and $\epsilon$, (1) and (2) imply

$$k' \cot k' r_0 = k' \cot \sqrt{k_0^2 + k^2}\, r_0 \doteqdot k' \cot k_0 r_0 \left(1 + \frac{k^2}{2k_0^2}\right)$$

$$= k' \cot\left(k_0 r_0 + \frac{1}{2}\frac{k^2 r_0}{k_0}\right) = k' \cot\left[(2n+1)\frac{\pi}{2} + \epsilon + \frac{1}{2}\frac{k^2 r_0}{k_0}\right]$$

So

$$k' \cot k' r_0 \approx - k'\left(\epsilon + \frac{1}{2}\frac{k_1^2 r_0}{k_0}\right). \tag{3}$$

The RHS of (1) for tiny $k$ is

$$k \cot(k r_0 + \delta_0(k)) = k \cot\delta_0(k) - \frac{k^2 r_0}{\sin^2\delta_0(k)}. \tag{4}$$

So we have

$$-k'\left(\epsilon + \frac{1}{2}\frac{k^2 r_0}{k_0}\right) = k \cot\delta_0(k) - \frac{k^2 r_0}{\sin^2\delta_0(k)}. \tag{5}$$

Thus if

$$k^2 \approx -\frac{2k_0\epsilon}{r_0} > 0 \tag{6}$$

then

$$\cot\delta_0(k) \approx 0 \tag{7}$$

which means that

$$\delta_0(k) = (2n+1)\frac{\pi}{2}. \tag{8}$$

Thus if $k^2 \approx -2k_0\epsilon/r_0$, then the s-wave x-section

$$\sigma_0(k) = \frac{4\pi}{k^2}\sin^2\delta_0(k) \tag{9}$$

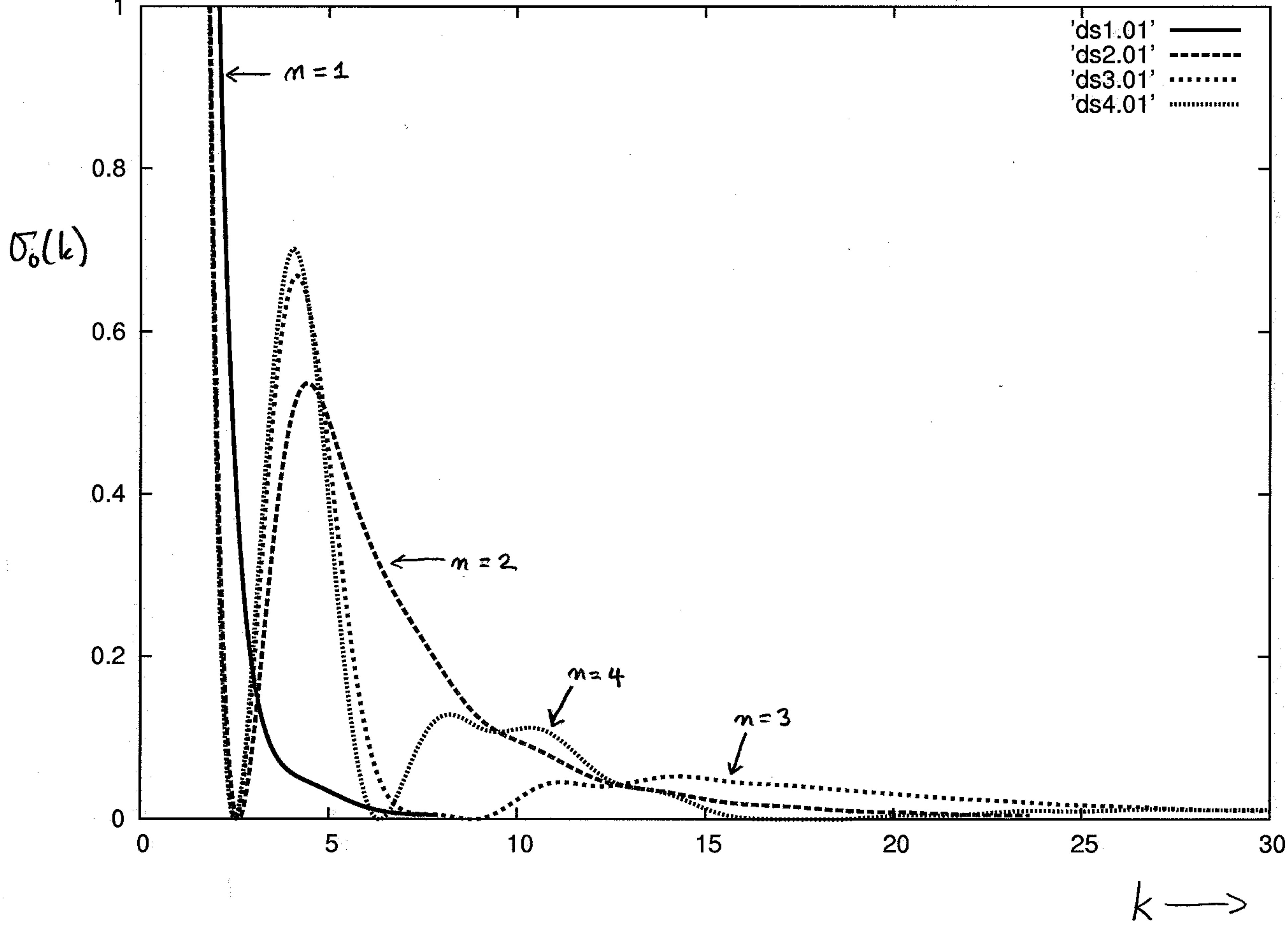
is maximal — there is a resonance at $k^2 = \frac{-2k_0\epsilon}{r_0}$.

3. Use a computer (or your fingers) to graph the total s-wave x-section $\sigma_0(k)$ as a function of $k$ for $k_0 r_0 = (2n+1)\pi/2 \pm \epsilon$. Do this for a few values of $n \leq 4$.

What follows is a computer program written in Fortran 90 and a composite plot of the s-wave x-sections $\sigma_0(k)$.

```
program swaves
  use defs; implicit none
  character(len=20)afile,dfile,zfile,sfile,ksfile
  character(len=1)::an
  character(len=3)::aE
  integer(i4b)::i,n
  real(dp)::x,r0,k,kp,kpp,k0,d,z,s,E
  afile = 'ScatteringLength'
  open(7,file=afile)
  do i = 1, 1000
     x = i*twopi/1000
     write(7,*) x, 1.0d0 - tan(x)/x
  end do
  close(7)
  write(6,*)'What is the radius of the sphere?'
  read(5,*) r0
  write(6,*)'How many bound states?'
  read(5,*)n
  write(6,*)'Again, how many bound states?'
  read(5,*)an
  write(6,*)'Energy margin E?'
  read(5,*)E
  write(6,*)'Again, energy margin E?'
  read(5,*)aE
  aE=trim(aE)
  if ( n == 0 ) then
     k0 = E
  else
     k0 = (2*(n-1)+1)*0.5d0*pi + E
  end if
  dfile = trim('s-wavePhaseshift')//an//aE
  sfile=trim('ds'//an//aE)
  ksfile=trim('k^2ds'//an//aE)
  open(8,file=dfile)
  open(10,file=sfile); open(11,file=ksfile)
  do i = 1, 1000
     k = 5.d0*k0*i/1000
     kp = sqrt(k0**2 + k**2)
     d = atan((k/kp)*tan(kp*r0)) - k*r0
     write(8,*) k, d
     write(10,*) k, 4.0d0*pi*(sin(d)/k)**2
     write(11,*) k, 4.0d0*pi*sin(d)**2
  end do
  close(8)
  close(10);close(11)
  zfile=trim('boundStates'//an//aE)
  open(9,file=zfile)
  do i = 1, 999
     kpp = k0*i/1000
     kp = sqrt(k0**2 - kpp**2)
     z = tan(kp*r0) + kp/kpp
     write(9,*) kpp, z
  end do
  close(9)
end program swaves
```

S-wave x-sections for $k_0 r_0 = [2(n-1)+1]\frac{\pi}{2} + 0.01$ for $n = 1, 2, 3, 4$.
(For $n = 0$, $\sigma_0(k)$ is too small to see on this scale.)



4. Use equations (27) and (33) of the class notes "Light and Atoms in SI Units" to derive Eq.(36) of those notes.

We set

$$f_{kr} = \left( \frac{\hbar}{2\epsilon_0 V \omega_k} \right)^{\frac{1}{2}}$$

so that

$$\vec{A}(\vec{x},t) = \sum_{kr} f_{kr} \left[ \epsilon_r a_r(k) e^{i(kx-\omega t)} + \epsilon_r^* a_r^\dagger(k) e^{-i(kx-\omega t)} \right].$$

The hamiltonian $H_{0F}$ is

$$H_{0F} = \int d^3x \left[ \frac{\epsilon_0}{2} \vec{E}(x,t)^2 + \frac{1}{2\mu_0} \vec{B}^2(x,t) \right]$$

in which

$$\vec{E} = -\dot{\vec{A}} \quad \text{and} \quad B = \nabla \times A.$$

So the $E^2$ part of $H_{0F}$ is

$$H_{0FE} = \int d^3x \frac{\epsilon_0}{2} \dot{A}^2$$

$$= \frac{\epsilon_0}{2} \int d^3x \sum_{kr} f_{kr} \left[ -i\omega \epsilon_r a_r e^{i(k\cdot x - \omega t)} + i\omega \epsilon^* a^\dagger e^{-i(k\cdot x - \omega t)} \right]$$

$$\times \sum_{k'r'} f_{k'r'} \left[ -i\omega' \epsilon_{r'} a_{r'} e^{i(k'x - \omega' t)} + i\omega' \epsilon^* a^\dagger e^{-i(k'\cdot x - \omega' t)} \right].$$

Now $\int d^3x \, e^{ikx + ik'\cdot x} = \delta_{k,-k'} V$ and

$$\int d^3x \; e^{ik\cdot x - ik'\cdot x} = \delta_{kk'} V$$

So $H_{OFE}$ is

$$H_{OFE} = \frac{\epsilon_0}{2} \sum_{k,r,r'} \frac{-\omega_k^2 \hbar V}{2\epsilon_0 V \omega_k} \vec{\epsilon}_r(k)\cdot\vec{\epsilon}_{r'}(-k)\, a_r(\vec{k})\, a_{r'}(-\vec{k}')\, e^{-2i\omega_k t}$$

$$+ \frac{\epsilon_0}{2} \sum_{krr'} \frac{-\omega_k^2 \hbar}{2\epsilon_0 \omega_k} \vec{\epsilon}_r^{\,*}(k)\cdot\vec{\epsilon}_{r'}^{\,*}(-\vec{k})\, a_r^{+}(k)\, a_{r'}^{+}(-k)\, e^{2i\omega_k t}$$

$$+ \frac{\epsilon_0}{2} \sum_{kr} \frac{\omega_k \hbar}{2\epsilon_0} a_r(k) a_r^{+}(k) + \frac{\omega_k \hbar}{2\epsilon_0} a_r^{+}(k) a_r(k) .$$

Obviously, we want the complicated sums to cancel similar terms in $H_{OFB}$ which involves

$$B = \nabla \times A = \sum_{kr} f_{kr} \left[ i k \times \epsilon_r(k)\, a_r(k)\, e^{i(k\cdot x - \omega t)} - i k \times \epsilon_r^{*}(k)\, a_r^{+}(k)\, e^{-i(k\cdot x - \omega t)} \right].$$

So

$$H_{OFB} = \frac{1}{2\mu_0} \int d^3x \, (\nabla \times A)^2$$

$$= \frac{1}{2\mu_0} \int d^3x \sum_{kr} f_{kr} \left[ i k \times \epsilon a e^{i(k\cdot x - \omega t)} - i k \times \epsilon^{*} a^{+} e^{-i(k\cdot x - \omega t)} \right]$$

$$\times \sum_{k'r'} f_{k'r'} \left[ i k' \times \epsilon' a' e^{i(k'\cdot x - \omega' t)} - i k' \times \epsilon'^{*} a'^{+} e^{-i(k'\cdot x - \omega' t)} \right].$$

So

$$H_{0FB} = \frac{1}{2\mu_0} \sum_{k r r'} \frac{\hbar}{2\epsilon_0 \omega_k} k\times\epsilon_r(k)\cdot k\times\epsilon_{r'}(-k)\, a_r(k)\, a_{r'}(-k)\, e^{-2i\omega_k t}$$

$$+ \frac{1}{2\mu_0} \sum_{k r r'} \frac{\hbar}{2\epsilon_0 \omega_k} k\times\epsilon^*(k)\cdot k\times\epsilon^*(-k)\, a_r^\dagger(k)\, a_{r'}^\dagger(-k)\, e^{2i\omega_k t}$$

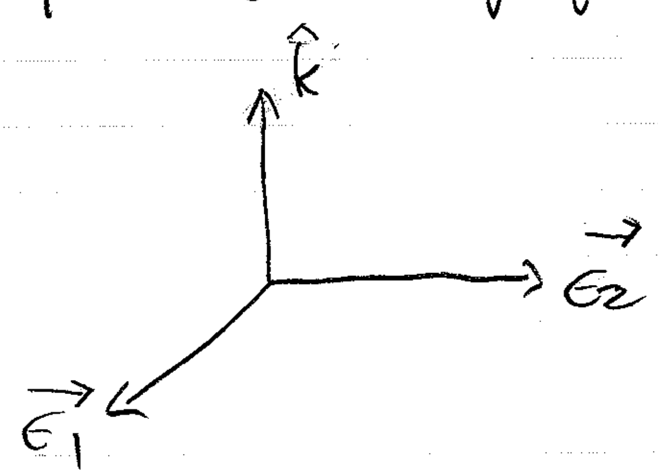
$$+ \frac{1}{2\mu_0} \sum_{k r} \frac{\hbar}{2\epsilon_0 \omega_k} k\times\epsilon\cdot k\times\epsilon^* \left[ a_r(k) a_r^\dagger(k) + a_r^\dagger(k) a_r(k) \right].$$

Again, we want to keep the simple terms and cancel the complicated ones.

The easy way to proceed is to choose polarization vectors $\vec{\epsilon}_r(\vec{k})$ that are real. They must be orthogonal to $\vec{k}$

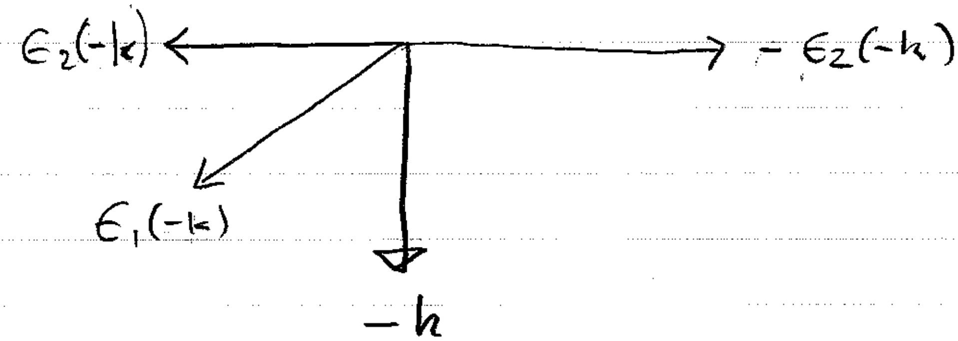
$$\vec{\epsilon}_r(\vec{k}) \cdot \vec{k} = 0$$

because of the Coulomb-gauge condition $\nabla \cdot A = 0$.



So
$$\hat{k} \times \epsilon_1 = \epsilon_2$$
$$\hat{k} \times \epsilon_2 = -\epsilon_1 .$$

Also we choose




So that
$$-\hat{k} \times \epsilon_1(-k) = \epsilon_2(-k) = -\epsilon_2(k)$$
$$-\hat{k} \times \epsilon_2(-k) = -\epsilon_1(-k) = -\epsilon_1(k)$$

With these choices of polarization vectors,

$$\epsilon_1(k) \cdot \epsilon_1(-k) = 1$$
$$\epsilon_1(k) \cdot \epsilon_2(-k) = 0$$
$$\epsilon_2(k) \cdot \epsilon_1(-k) = 0$$
$$\epsilon_2(k) \cdot \epsilon_2(-k) = -1 .$$

Now the electric field energy is

$$H_{OFE} = \frac{1}{4}\sum_k -\hbar\omega_k \left[a_1(k)a_1(-k) - a_2(k)a_2(-k)\right] e^{-2i\omega_k t}$$

$$+ \frac{1}{4}\sum_k -\hbar\omega_k \left[a_1(k)a_1(-k) - a_2(k)a_2(-k)\right] e^{2i\omega_k t}$$

$$+ \frac{1}{4}\sum_{k,n} \hbar\omega_k \left[a_n(k)a_n^{\dagger}(k) + a_n^{\dagger}(k)a_n(k)\right].$$

The magnetic field energy involves

$$\hat{k}\times\epsilon_1(k)\cdot\hat{k}\times\epsilon_1(-k) = \epsilon_2(k)\cdot(-\epsilon_2(-k)) = \epsilon_2(k)\cdot\epsilon_2(k) = 1$$

$$\hat{k}\times\epsilon_1(k)\cdot\hat{k}\times\epsilon_2(-k) = \epsilon_2(k)\cdot\epsilon_1(-k) = \epsilon_2(k)\cdot\epsilon_1(k) = 0$$

$$\hat{k}\times\epsilon_2(k)\cdot\hat{k}\times\epsilon_1(-k) = -\epsilon_1(k)\cdot(-\epsilon_2(-k)) = 0$$

$$\hat{k}\times\epsilon_2(k)\cdot\hat{k}\times\epsilon_2(-k) = -\epsilon_1(k)\cdot\epsilon_1(-k) = -\epsilon_1(k)\cdot\epsilon_1(k) = -1.$$

So since $\epsilon_0\mu_0 = 1/c^2$ and $\omega_k = kc$,

$$H_{OFB} = \frac{c^2}{4}\sum_k \frac{\hbar k^2}{\omega_k}\left[a_1(k)a_1(-k) - a_2(k)a_2(-k)\right] e^{-2i\omega_k t}$$

$$+ \frac{1}{4}\sum_k \hbar\omega_k \left[a_1^{\dagger}(k)a_1^{\dagger}(-k) - a_2^{\dagger}(k)a_2^{\dagger}(-k)\right] e^{2i\omega_k t}$$

$$+ \frac{1}{4}\sum_{kn} \hbar\omega_k \left[a_n(k)a_n^{\dagger}(k) + a_n^{\dagger}(k)a_n(k)\right].$$

We see that the unwanted terms exactly cancel

$$H = H_{OFE} + H_{OFB}$$

$$= \frac{1}{2} \sum_{k\nu} \hbar\omega_k \left[ a_\nu(k) a_\nu^\dagger(k) + a_\nu^\dagger(k) a_\nu(k) \right]$$

$$= \sum_{k\nu} \hbar\omega_k \left[ a_\nu^\dagger(k) a_\nu(k) + \frac{1}{2} \right]$$

since

$$[a_\nu(k), a_{\nu'}^\dagger(k')] = \delta_{\nu\nu'} \delta_{kk'} .$$