1. For an eigenstate $|E\rangle$ of the hamiltonian

$$H = \frac{\mathbf{p}^2}{2m} + g^2\left(x^4 + y^4 + z^4\right) \tag{1}$$

what is the relation between the mean value of the kinetic energy and that of the potential energy $V = g^2\left(x^4 + y^4 + z^4\right)$?

The potential $V(\vec{r}) = g^2(x^4 + y^4 + z^4)$ is homogeneous of degree $n = 4$ since

$$V(\alpha\vec{r}) = \alpha^4 V(\vec{r}).$$

The quantum virial theorem says that in an eigenstate $|E\rangle$ of $H = K + V$

$$\langle E|K|E\rangle = \langle E|\frac{\vec{p}^{\,2}}{2m}|E\rangle = \frac{n}{2}\langle E|V|E\rangle$$

$$= 2\langle E|V|E\rangle.$$

So the desired relation is

$$\langle E|K|E\rangle = 2\langle E|V|E\rangle.$$

2. Same problem but with

$$V = g^2\left(x^n + y^n + z^n\right). \tag{2}$$

Using the method of problem 1, we see that V is homogeneous of degree n and so that

$$\langle E|K|E\rangle = \frac{n}{2}\langle E|V|E\rangle.$$

3. Find the eigenvalues and eigenvectors of

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{3}$$

Consider an electron in the state

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha|+\rangle + \beta|-\rangle. \tag{4}$$

If we measure $S_2 = (\hbar/2)\sigma_2$, what is the probability of the result $\hbar/2$?

$$0 = \begin{vmatrix} -e & -i \\ i & -e \end{vmatrix} = e^2 - 1, \quad \text{So} \quad e = \pm 1.$$

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \pm\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -ib \\ ia \end{pmatrix}$$

So $ia = \pm b$ and $-ib = \pm a$, i.e., $b = \pm i a$

So $\chi_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix}$. If $\psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, then the

probability of $S_y = \frac{\hbar}{2}$ is

$$P = |\chi_+^\dagger \psi|^2 = \left|\frac{1}{\sqrt{2}}(1, -i)\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right|^2 = \frac{1}{2}|\alpha - i\beta|^2.$$

4. The spin-dependent part of the hamiltonian H of an electron and a positron in a uniform magnetic field B in the z-direction is

$$H = A\boldsymbol{S}_e \cdot \boldsymbol{S}_p + \frac{eB}{mc}\left(S_{ze} - S_{zp}\right). \tag{5}$$

Consider the state $|+,-\rangle = |+_e, -_p\rangle$ in which the electron is "spin up" and the proton "spin down." (**a**): Is this state an eigenstate of H in the limit $A \to 0$ with $eB/mc \neq 0$? If yes, what is its energy eigenvalue? If no, what is the mean value of H in this state? (**b**): Same problem in the opposite limit, $eB/mc \to 0$ with $A \neq 0$.

$$H = A \vec{S}^{(e^-)} \cdot \vec{S}^{(e^+)} + \left(\frac{eB}{mc}\right)\left(S_z^{(e^-)} - S_z^{(e^+)}\right),$$

(a) $$H_{A=0} = \frac{eB}{mc}\left(S_z^{(e^-)} - S_z^{(e^+)}\right)$$

$$H_{A=0}|+,-\rangle = \left(\frac{eB}{mc}\right)\left(S_z^{(e^-)}|+-\rangle - S_z^{(e^+)}|+-\rangle\right)$$

$$= \left(\frac{eB}{mc}\right)\left(\frac{\hbar}{2}|+-\rangle + \frac{\hbar}{2}|+-\rangle\right)$$

$$= \frac{e\hbar B}{mc}|+-\rangle.$$ So, yes, $|+-\rangle \doteq \chi_+^{(e^-)}\chi_-^{(e^+)}$ is an evec of $H_{A=0}$ with eval $e\hbar B/mc$.

(b) $H_{B=0} = A\vec{S}^{(e^-)} \cdot \vec{S}^{(e^+)}$. No, $|+-\rangle$ is not an evec of $H_{B=0}$.

$$\langle +-|H_{B=0}|+-\rangle = A\langle +-|S_z^{(e^-)} S_z^{(e^+)}|+-\rangle = -\frac{\hbar^2}{4}A.$$

5. If $\boldsymbol{S}$ is the spin operator of a massive, spin-one particle, what are the matrix elements of the operators

$$S_z(S_z - \hbar)(S_z + \hbar) \quad \text{and} \quad S_y(S_y - \hbar)(S_y + \hbar) \tag{6}$$

between its various states?

The evals of S_z are 0 and $\pm\hbar$ for spin 1. The evals of S_x are 0 and $\pm\hbar$ for $s=1$.

Thus by problem (7a) on p. 61 all matrix elements of both compound operators

$$0 = S_z(S_z+\hbar)(S_z-\hbar) \quad \text{and} \quad S_x(S_x+\hbar)(S_x-\hbar) = 0$$

must vanish. These operators are identically zero as operators.