Example: the addition of two $j = \frac{1}{2}$'s.

We use
$$J_\pm |jm\rangle = \hbar \sqrt{(j \mp m)(j+1 \pm m)}\ |j, m\pm 1\rangle.$$

The state of highest m is $|j_1 = \frac{1}{2}, m_1 = \frac{1}{2}\rangle |j_2 = \frac{1}{2}, m_2 = \frac{1}{2}\rangle$
$= |+\rangle|+\rangle$ which has $j = 1$

$|1,1\rangle = |+\rangle|+\rangle$. Now we apply J_-

$$J_- |1,1\rangle = \hbar\sqrt{(1+1)(1+1-1)}\ |1,0\rangle = \hbar\sqrt{2}\ |1,0\rangle$$

$$= (J_{1-} + J_{2-})|+\rangle|+\rangle = (J_{1-}|+\rangle)|+\rangle + |+\rangle(J_{2-}|+\rangle)$$

$$= \hbar\sqrt{(\tfrac{1}{2}+\tfrac{1}{2})(\tfrac{1}{2}+1-\tfrac{1}{2})}\ |-\rangle|+\rangle + \hbar|+\rangle\sqrt{(\tfrac{1}{2}+\tfrac{1}{2})(\tfrac{3}{2}-\tfrac{1}{2})}\ |+\rangle|-\rangle$$

$$= \hbar\left(|-\rangle|+\rangle + |+\rangle|-\rangle\right). \qquad \text{So}$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle|-\rangle + |-\rangle|+\rangle\right) = \frac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right).$$

Clearly, $|1,-1\rangle = |--\rangle = |-\rangle|-\rangle$.

The other state has $j=0$ and $m=0$. It must be a linear combination of $|+-\rangle$ and $|-+\rangle$ that is orthogonal to $|1,0\rangle = (|+-\rangle + |-+\rangle)/\sqrt{2}$ and the other $|1m\rangle$ states. By convention, we choose
$$|0,0\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right).$$