

Addition of Angular Momenta

Suppose, our system has two angular momenta \vec{J}_1 and \vec{J}_2 . Then one set of states is the e-vecs of \vec{J}_1^2 , \vec{J}_2^2 , J_{1z} , J_{2z} , & $J_{1z} + J_{2z}$. These states are the direct products

$$|j_1, j_2, m_1, m_2\rangle = |j_1, m_1\rangle \otimes |j_2, m_2\rangle \quad (1)$$

like the states $|l, m \pm \frac{1}{2}\rangle$ that diagonalize the important parts $H_0 + H_1$ of the H of an H-atom in a constant magnetic field.

We have

$$\begin{aligned} J_1^2 |j_1, j_2, m_1, m_2\rangle &= (J_1^2 |j_1, m_1\rangle) \otimes |j_2, m_2\rangle \\ &= \hbar^2 j_1(j_1+1) |j_1, m_1\rangle \otimes |j_2, m_2\rangle \\ &= \hbar^2 j_1(j_1+1) |j_1, j_2, m_1, m_2\rangle \quad (2) \end{aligned}$$

$$\begin{aligned} J_2^2 |j_1, j_2, m_1, m_2\rangle &= |j_1, m_1\rangle \otimes (J_2^2 |j_2, m_2\rangle) \\ &= \hbar^2 j_2(j_2+1) |j_1, j_2, m_1, m_2\rangle \quad (3) \end{aligned}$$

$$\begin{aligned} J_{1z} |j_1, j_2, m_1, m_2\rangle &= (J_{1z} |j_1, m_1\rangle) \otimes |j_2, m_2\rangle \\ &= \hbar m_1 |j_1, j_2, m_1, m_2\rangle \quad (4) \end{aligned}$$

$$\begin{aligned}
 J_{2z} |j_1, j_2, m_1, m_2\rangle &= |j_1, m_1\rangle \otimes (J_{2z} |j_2, m_2\rangle) \\
 &= \hbar m_2 |j_1, j_2, m_1, m_2\rangle \quad (5)
 \end{aligned}$$

$$\begin{aligned}
 (J_{1z} + J_{2z}) |j_1, j_2, m_1, m_2\rangle &= J_{1z} |j_1, j_2, m_1, m_2\rangle + J_{2z} |j_1, j_2, m_1, m_2\rangle \\
 &= \hbar (m_1 + m_2) |j_1, j_2, m_1, m_2\rangle. \quad (6)
 \end{aligned}$$

These states $|j_1, j_2, m_1, m_2\rangle$ form a complete orthonormal set of states for the system of \vec{J}_1 and \vec{J}_2 .

$$\langle j_1, j_2, m_1, m_2 | j_1, j_2, m_1', m_2' \rangle = \delta_{m_1, m_1'} \delta_{m_2, m_2'}. \quad (7)$$

But we also could find e-vecs of $\vec{J}_1^2, \vec{J}_2^2,$

$$J_z = J_{1z} + J_{2z}, \quad (8)$$

and

$$\vec{J}^2 = (\vec{J}_1 + \vec{J}_2)^2. \quad (9)$$

First, let's check that \vec{J} is an angular momentum, that is, that it satisfies the rule

$$[J_i, J_j] = \hbar \sum_{k=1}^3 \epsilon_{ijk} J_k. \quad (10)$$

Well, since the two J 's are independent

$$[J_{1i}, J_{2j}] = 0 \quad (11)$$

we have

$$\begin{aligned} [J_i, J_j] &= [J_{1i} + J_{2i}, J_{1j} + J_{2j}] \\ &= [J_{1i}, J_{1j}] + [J_{2i}, J_{2j}] \\ &= i\hbar \sum_K \epsilon_{ijk} J_{1K} + i\hbar \sum_K \epsilon_{ijk} J_{2K} \\ &= i\hbar \sum_K \epsilon_{ijk} (J_{1K} + J_{2K}) \\ &= i\hbar \sum_K \epsilon_{ijk} J_K. \end{aligned} \quad (12)$$

So $\vec{J} = \vec{J}_1 + \vec{J}_2$ is an angular momentum.
Since it is, we know that

$$[J^2, J_z] = 0 \quad (13)$$

which is just the statement that J^2 is a scalar.

Now note that by (11)

$$\begin{aligned} \vec{J}^2 &= (\vec{J}_1 + \vec{J}_2)^2 = \vec{J}_1^2 + \vec{J}_2^2 + 2\vec{J}_1 \cdot \vec{J}_2 \quad (14) \\ &= \vec{J}_1^2 + \vec{J}_2^2 + 2J_{1z}J_{2z} + 2J_{1x}J_{2x} + 2J_{1y}J_{2y} \\ &= \vec{J}_1^2 + \vec{J}_2^2 + 2J_{1z}J_{2z} + (J_{1x} + iJ_{1y})(J_{2x} - iJ_{2y}) \\ &\quad + (J_{1x} - iJ_{1y})(J_{2x} + iJ_{2y}) \quad (15) \end{aligned}$$

so

$$\vec{J}^2 = J_1^2 + J_2^2 + 2J_{1z}J_{2z} + J_{1+}J_{2-} + J_{1-}J_{2+}, \quad (16)$$

$$\text{Thus } [\vec{J}^2, J_1^2] = [\vec{J}^2, J_2^2] = 0. \quad (17)$$

So \vec{J}^2 , J_z , J_1^2 , and J_2^2 all commute with each other. We can find simultaneous e-vecs $|j_1 j_2 j_m\rangle$

$$\vec{J}_1^2 |j_1 j_2 j_m\rangle = \hbar^2 j_1(j_1+1) |j_1 j_2 j_m\rangle$$

$$\vec{J}_2^2 |j_1 j_2 j_m\rangle = \hbar^2 j_2(j_2+1) |j_1 j_2 j_m\rangle$$

$$\vec{J}^2 |j_1 j_2 j_m\rangle = \hbar^2 j(j+1) |j_1 j_2 j_m\rangle$$

$$J_z |j_1 j_2 j_m\rangle = \hbar m |j_1 j_2 j_m\rangle \quad (18)$$

in which

$$-j \leq m \leq j. \quad (19)$$

Note, however, that $[\vec{J}^2, J_{1z}] \neq 0 \neq [\vec{J}^2, J_{2z}]$ because, e.g., by (16) and $[J_z, J_{\pm}] = \pm \hbar J_{\pm}$ (3.5.6b)

$$[J^2, J_{1z}] = [J_{1+}, J_{1z}]J_{2-} + [J_{1-}, J_{1z}]J_{2+}$$

$$= -\hbar J_{1+}J_{2-} + \hbar J_{1-}J_{2+} \neq 0. \quad (20)$$

So we can't add J_{1z} or J_{2z} to J^2, J_z, J_1^2 & J_2^2 .

So we have two complete orthonormal sets of states — the $(2j_1+1)(2j_2+1)$ states

$$|j_1, j_2, m_1, m_2\rangle \quad (21)$$

and the states

$$|j_1, j_2, j, m\rangle. \quad (22)$$

Among the states (21) the one with the highest value of J_z is the state

$$|j_1, j_2, m_1=j_1, m_2=j_2\rangle = |j_1, j_2, j_1, j_2\rangle \quad (23)$$

$$\begin{aligned} J_z |j_1, j_2, j_1, j_2\rangle &= (J_{z1} + J_{z2}) |j_1, j_2, j_1, j_2\rangle \\ &= \hbar (j_1 + j_2) |j_1, j_2, j_1, j_2\rangle. \end{aligned} \quad (24)$$

So some linear combination of the states (22) should have the same e-value of J_z

$$\begin{aligned} J_z |j_1, j_2, j, m\rangle &= \hbar (j_1 + j_2) |j_1, j_2, j, m\rangle \\ &= \hbar m |j_1, j_2, j, m\rangle. \end{aligned} \quad (25)$$

and this should be the highest possible e-value of J_z . But the state $|j_1, j_2, j, m\rangle$ with the highest m is

$$|j_1, j_2, j, j\rangle \quad (26)$$

for the highest value of j . Thus the highest

value of j must be

$$j = j_1 + j_2. \quad (27)$$

What is the minimum value of j ?
Call it j_0 . Then the states $|j, j_2, j_m\rangle$
consist of

$$2j_0 + 1 \quad |j, j_2, j_0, m\rangle's$$

$$2(j_0 + 1) + 1 \quad |j, j_2, j_0 + 1, m\rangle's \quad (28)$$

up to

$$2(j_1 + j_2) + 1 \quad |j, j_2, j_1 + j_2, m\rangle's.$$

There must be as many $|j, j_2, m\rangle$ states
as $|j, j_2, m_1, m_2\rangle$ states. So

$$\sum_{j=j_0}^{j_1+j_2} 2j+1 = (2j_1+1)(2j_2+1)$$

$$= \sum_{j=0}^{j_1+j_2} 2j+1 - \sum_{j=0}^{j_0-1} 2j+1. \quad (29)$$

Use $\sum_{n=0}^N n = \frac{N}{2}(N+1)$ and $\sum_{n=0}^N 1 = N+1$ (30)

$$\text{So find } (2j_1+1)(2j_2+1) =$$

$$2 \frac{(j_1+j_2)(j_1+j_2+1)}{2} + j_1+j_2+1 - 2 \frac{j_0(j_0-1)}{2} - j_0. \quad (31)$$

So j_0 is a solution of

$$(2j_1+1)(2j_2+1) = (j_1+j_2)(j_1+j_2+1) + j_1+j_2+1 - j_0^2 \quad (31)$$

$$\begin{aligned} j_0^2 &= (j_1+j_2)(j_1+j_2+1) + j_1+j_2+1 - (2j_1+1)(2j_2+1) \\ &= j_1^2 + j_2^2 + 2j_1j_2 + 2j_1 + 2j_2 + 1 - 4j_1j_2 - 2j_1 - 2j_2 - 1 \\ &= (j_1 - j_2)^2, \end{aligned} \quad (32)$$

That is, the minimum value of j must be

$$j = |j_1 - j_2|. \quad (33)$$

So the states $|j_1, j_2, j, m\rangle$ run from

$$|j_1, j_2, |j_1 - j_2|, m\rangle \text{ to } |j_1, j_2, j_1 + j_2, m\rangle. \quad (34)$$

Since

$$|j_1, j_2, m_1, m_2\rangle \times |j_1, j_2, m_1, m_2\rangle \quad (35)$$

we have

$$|j_1, j_2, j, m\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1, j_2, m_1, m_2\rangle \times |j_1, j_2, m_1, m_2\rangle |j_1, j_2, j, m\rangle. \quad (36)$$

The numbers

$$\langle j_1, j_2, m_1, m_2 | j, j_z, m \rangle \quad (37)$$

are the Clebsch-Gordan coefficients (CGC's).

They vanish unless $m_1 + m_2 = m$

since

$$0 = J_z - J_{1z} - J_{2z} \quad (38)$$

whence

$$\begin{aligned} 0 &= \langle j_1, j_2, m_1, m_2 | J_z - J_{1z} - J_{2z} | j, j_z, m \rangle \\ &= \hbar (m - m_1 - m_2) \langle j_1, j_2, m_1, m_2 | j, j_z, m \rangle. \end{aligned} \quad (39)$$

So

$$m - m_1 - m_2 \neq 0 \Rightarrow \langle j_1, j_2, m_1, m_2 | j, j_z, m \rangle = 0. \quad (40)$$

We easily may find two recursion relations for the CGC's by using (3.5.39-40)

$$\langle j_1, m_1 \pm 1 | J_{\pm} | j, m \rangle = \sqrt{(j \mp m)(j \pm m \pm 1)} \hbar | j, m \pm 1 \rangle \quad (41)$$

We apply $J_{\pm} = J_{1\pm} + J_{2\pm}$ to Eq. (36):

$$\begin{aligned}
J_{\pm} |j_1, j_2, m\rangle &= \hbar \sqrt{(j_1 \mp m)(j_1 \pm m + 1)} |j_1, j_2, m \pm 1\rangle \\
&= (J_{1\pm} + J_{2\pm}) \sum_{m_1, m_2} |j_1, j_2, m_1, m_2\rangle \times |j_1, j_2, m_1, m_2\rangle \\
&= \sum_{m_1, m_2} \left[(J_{1\pm} |j_1, m_1\rangle) \otimes |j_2, m_2\rangle + |j_1, m_1\rangle \otimes (J_{2\pm} |j_2, m_2\rangle) \right] \\
&\quad \times \langle j_1, j_2, m_1, m_2 | j_1, j_2, m \rangle \\
&= \hbar \sum_{m_1, m_2} \left[\sqrt{(j_1 \mp m_1)(j_1 \pm m_1 + 1)} |j_1, j_2, m_1 \pm 1, m_2\rangle \right. \\
&\quad \left. + \sqrt{(j_2 \mp m_2)(j_2 \pm m_2 + 1)} |j_1, j_2, m_1, m_2 \pm 1\rangle \right] \\
&\quad \times \langle j_1, j_2, m_1, m_2 | j_1, j_2, m \rangle, \quad (42)
\end{aligned}$$

We now take the inner product of this ket equation with $|j_1, j_2, m_1, m_2\rangle$ using the or rule

$$0 = \langle j_1, j_2, m_1, m_2 | j_1, j_2, m_1' \pm 1, m_2' \rangle \quad \text{unless } m_1 = m_1' \pm 1 \text{ and } m_2 = m_2' \quad (43)$$

and

$$0 = \langle j_1, j_2, m_1, m_2 | j_1, j_2, m_1', m_2' \pm 1 \rangle \quad \text{unless } m_1 = m_1' \text{ and } m_2 = m_2' \pm 1. \quad (44)$$

Thus

$$\begin{aligned} & \sqrt{(j_1 \mp m)(j_1 \pm m + 1)} \langle j_1, j_2, m_1, m_2 | j_1, j_2, m \pm 1 \rangle \\ &= \sqrt{(j_1 \mp (m_1 \mp 1))(j_1 \pm (m_1 \mp 1) + 1)} \langle j_1, j_2, m_1 \mp 1, m_2 | j_1, j_2, m \rangle \\ &+ \sqrt{(j_2 \mp (m_2 \mp 1))(j_2 \pm (m_2 \mp 1) + 1)} \langle j_1, j_2, m_1, m_2 \mp 1 | j_1, j_2, m \rangle \quad (45) \end{aligned}$$

$$\begin{aligned} &= \sqrt{(j_1 \mp m_1 + 1)(j_1 \pm m_1)} \langle j_1, j_2, m_1 \mp 1, m_2 | j_1, j_2, m \rangle \\ &+ \sqrt{(j_2 \mp m_2 + 1)(j_2 \pm m_2)} \langle j_1, j_2, m_1, m_2 \mp 1 | j_1, j_2, m \rangle. \quad (46) \end{aligned}$$

For the simple cases most often of interest, these relations are pretty easy to use. For instance, an H-atom in a magnetic field has e-vecs whose angular parts are

$$|l, \frac{1}{2}, m \pm \frac{1}{2}\rangle \quad (47)$$

when we neglect the \vec{A}^2 term. We can combine these states to form e-vecs of J^2 and J_z . The top state with $j = l + \frac{1}{2}$ and $m = j$ is

$$|l, \frac{1}{2}, l + \frac{1}{2}\rangle = |l, \frac{1}{2}, l, \frac{1}{2}\rangle. \quad (48)$$

On it J_- gives by (41)

$$\begin{aligned} J_- |l, \frac{1}{2}, l + \frac{1}{2}\rangle &= \sqrt{(l + \frac{1}{2} + l + \frac{1}{2})(l + \frac{1}{2} - l - \frac{1}{2} + 1)} \hbar |l, \frac{1}{2}, l - \frac{1}{2}\rangle \\ &= \hbar \sqrt{2l + 1} |l, \frac{1}{2}, l - \frac{1}{2}\rangle \quad (49) \end{aligned}$$

$$\begin{aligned}
&= (J_{1-} + J_{2-}) |l \frac{1}{2} l \frac{1}{2}\rangle \\
&= (L_- + S_-) |l \frac{1}{2} l \frac{1}{2}\rangle \\
&= (L_- |l l\rangle) \otimes |l \frac{1}{2} \frac{1}{2}\rangle + |l l\rangle \otimes (S_- |l \frac{1}{2} \frac{1}{2}\rangle) \\
&= \hbar \sqrt{2l} |l l-1\rangle \otimes |l \frac{1}{2} \frac{1}{2}\rangle + |l l\rangle \otimes \hbar \sqrt{l+1} |l \frac{1}{2} -\frac{1}{2}\rangle \quad (42)
\end{aligned}$$

So

$$\sqrt{2l+1} |l \frac{1}{2} l + \frac{1}{2} l - \frac{1}{2}\rangle = \sqrt{2l} |l \frac{1}{2} l - \frac{1}{2}\rangle + |l \frac{1}{2} l - \frac{1}{2}\rangle \quad (43)$$

or

$$|l \frac{1}{2}, l + \frac{1}{2} l - \frac{1}{2}\rangle = \sqrt{\frac{2l}{2l+1}} |l \frac{1}{2} l - \frac{1}{2}\rangle + \frac{1}{\sqrt{2l+1}} |l \frac{1}{2} l - \frac{1}{2}\rangle \quad (44)$$

which is normalized.

The state $|l \frac{1}{2}, l - \frac{1}{2}, l - \frac{1}{2}\rangle$ must be orthogonal to $|l \frac{1}{2}, l + \frac{1}{2}, l - \frac{1}{2}\rangle$:

$$|l \frac{1}{2} l - \frac{1}{2} l - \frac{1}{2}\rangle = \frac{1}{\sqrt{2l+1}} |l \frac{1}{2} l - \frac{1}{2}\rangle - \sqrt{\frac{2l}{2l+1}} |l \frac{1}{2}, l, -\frac{1}{2}\rangle \quad (45)$$

apart from an arbitrary over-all factor of ± 1 .

The next step is to apply

$$J_- = L_- + S_- \text{ to both } |l \frac{1}{2}, l + \frac{1}{2}, l - \frac{1}{2}\rangle \text{ \& } |l \frac{1}{2}, l - \frac{1}{2}, l - \frac{1}{2}\rangle. \quad (46)$$

This procedure will relate all the states

$$|l, \frac{1}{2}, l + \frac{1}{2}, m\rangle \quad \text{for} \quad -l - \frac{1}{2} \leq m \leq l + \frac{1}{2} \quad (47)$$

and

$$|l, \frac{1}{2}, l - \frac{1}{2}, m\rangle \quad \text{for} \quad -l + \frac{1}{2} \leq m \leq l - \frac{1}{2} \quad (48)$$

to the states $|l, \frac{1}{2}, m, \pm \frac{1}{2}\rangle$ for $-l \leq m \leq l$. (49)

More generally, to add $\vec{J}_1 + \vec{J}_2$, one starts with the top state

$$|j_1, j_2, j_1 + j_2, j_1 + j_2\rangle = |j_1, j_2, j_1, j_2\rangle \quad (50)$$

where $j \uparrow m \uparrow$ $m_1 \uparrow m_2 \uparrow$

and applies $J_- = J_{1-} + J_{2-}$ using the rule

$$J_- |j, m\rangle = \hbar \sqrt{(j+m)(j-m+1)} |j, m-1\rangle. \quad (51)$$

We get

$$J_- |j_1, j_2, j_1 + j_2, j_1 + j_2\rangle = \hbar \sqrt{(2j_1 + 2j_2)} |j_1, j_2, j_1 + j_2 - 1, j_1 + j_2 - 1\rangle$$

$$= (J_{1-} |j_1, j_1\rangle) \otimes |j_2, j_2\rangle + |j_1, j_1\rangle \otimes (J_{2-} |j_2, j_2\rangle)$$

$$= \hbar \sqrt{2j_1} |j_1, j_1 - 1\rangle |j_2, j_2\rangle + |j_1, j_1\rangle \hbar \sqrt{2j_2} |j_2, j_2 - 1\rangle$$

$$= \hbar \left(\sqrt{2j_1} |j_1, j_2, j_1 - 1, j_2\rangle + \sqrt{2j_2} |j_1, j_2, j_1, j_2 - 1\rangle \right)$$

(52)

So

$$|j_1, j_2, j_1 + j_2, j_1 + j_2 - 1\rangle = \sqrt{\frac{j_1}{j_1 + j_2}} |j_1, j_2, j_1 - 1, j_2\rangle + \sqrt{\frac{j_2}{j_1 + j_2}} |j_1, j_2, j_1, j_2 - 1\rangle \quad (53)$$

The orthogonal state is

$$|j_1, j_2, j_1 + j_2 - 1, j_1 + j_2 - 1\rangle = \sqrt{\frac{j_2}{j_1 + j_2}} |j_1, j_2, j_1 - 1, j_2\rangle - \sqrt{\frac{j_1}{j_1 + j_2}} |j_1, j_2, j_1, j_2 - 1\rangle \quad (54)$$

to within a phase factor conventionally chosen as ± 1 .

One continues applying J_- to successive $|j_1, j_2, j_m\rangle$ states until one has expressed all $(2j_1 + 1)(2j_2 + 1)$ of them in terms of the $|j_1, j_2, m_1, m_2\rangle$ states.