20. We are to add angular momenta $j_1 = 1$ and $j_2 = 1$ to form $j = 2, 1,$ and 0 states. Using either the ladder operator method or the recursion relation, express all (nine) $\{j, m\}$ eigenkets in terms of $|j_1 j_2; m_1 m_2\rangle$.

(3-20) $|j=2, m=2\rangle = |2,2\rangle = |+,+\rangle$

$J_-|2,2\rangle = \hbar\sqrt{(2+2)(2-2+1)}\,|2,1\rangle = 2\hbar|2,1\rangle$

$= (J_{1-} + J_{2-})|++\rangle = \hbar\sqrt{(1+1)(1-1+1)}\,|0+\rangle + \hbar\sqrt{2}\,|+0\rangle$

So

$$|j=2, m=1\rangle = \frac{1}{\sqrt{2}}\left(|+0\rangle + |0+\rangle\right)$$

$J_-|2,1\rangle = \hbar\sqrt{3\cdot(2-1+1)}\,|2,0\rangle$

$= (J_{1-} + J_{2-})|2,1\rangle = \frac{1}{\sqrt{2}}(J_{1-} + J_{2-})\left(|+0\rangle + |0+\rangle\right)$

$= \frac{\hbar}{\sqrt{2}}\left[\sqrt{(1+1)(1-1+1)}\,|00\rangle + \sqrt{1\cdot(1+1)}\,|+-\rangle + \sqrt{2}\,|-+\rangle + \sqrt{2}\,|00\rangle\right]$

So

$$|j=2, m=0\rangle = \frac{1}{2\sqrt{3}}\left(\sqrt{2}\,|+-\rangle + 2\sqrt{2}\,|00\rangle + \sqrt{2}\,|-+\rangle\right)$$

$$= \frac{1}{\sqrt{6}}\left(|+-\rangle + 2|00\rangle + |-+\rangle\right).$$

$|j=2, m=-2\rangle = |2,-2\rangle = |-,-\rangle$

$J_+|2,-2\rangle = \hbar\sqrt{(2+2)(2-2+1)}\,|2,-1\rangle$

$= (J_{1+} + J_{2+})|2,-2\rangle = (J_{1+} + J_{2+})|-,-\rangle$

$= \hbar\left(\sqrt{(1+1)(1-1+1)}\,|0,-\rangle + \sqrt{2}\,|-,0\rangle\right)$ So

$$|j=2, m=-1\rangle = \frac{1}{\sqrt{2}}\left(|0,-\rangle + |-,0\rangle\right).$$

Now $|j=1, m=1\rangle = x|+0\rangle + y|0+\rangle$ and

$$0 = \langle j=2, m=1 | j=1, m=1\rangle = \frac{1}{\sqrt{2}}\left(\langle +0| + \langle 0+|\right)\left(x|+0\rangle + y|0+\rangle\right)$$

So $x+y=0$. Thus we may take

$$|j=1, m=1\rangle = \frac{1}{\sqrt{2}}\left(|+0\rangle - |0+\rangle\right) \equiv |1,1\rangle$$

in which the over-all phase is set by a convention of which I am ignorant.

$$J_-|j=1, m=1\rangle = \hbar\sqrt{(1+1)(1-1+1)}\,|1,0\rangle = \hbar\sqrt{2}\,|1,0\rangle$$

$$= (J_{1-} + J_{2-})\frac{1}{\sqrt{2}}\left(|+0\rangle - |0+\rangle\right)$$

$$= \frac{\hbar}{\sqrt{2}}\Big(\sqrt{(1+1)(1-1+1)}\,|00\rangle + \sqrt{(1+0)(1-0+1)}\,|+-\rangle$$
$$- \sqrt{2}|-+\rangle - \sqrt{2}\,|00\rangle\Big) = \hbar\left(|+-\rangle - |-+\rangle\right)$$

So

$$|j=1, m=0\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right).$$

And $|j=1, m=-1\rangle = \frac{1}{\sqrt{2}}\left(|0,-\rangle - |-,0\rangle\right)$

apart from a conventional phase factor.

Finally $|j=0, m=0\rangle = x|+-\rangle + y|00\rangle + z|-+\rangle$ and

$$0 = \langle 2,0|00\rangle = \left(\langle +-| + 2\langle 00| + \langle -+|\right)\left(x|+-\rangle + y|00\rangle + z|-+\rangle\right)$$

$0 = x + 2y + z$ Also $0 = \langle 1,0|0,0\rangle = \left(\langle +-| - \langle -+|\right)\left(x|+-\rangle + y|00\rangle + z|-+\rangle\right)$

or $0 = x - z$ So $z = x$ and $y = -x$. So

$$|j=0, m=0\rangle = \frac{1}{\sqrt{3}}\left(|+,-\rangle - |0,0\rangle + |-+\rangle\right).$$