Absorptive Scattering

For scattering off of a spherically symmetric potential $V(r)$, our formula was

$$\psi_k(r) = -\sum i^\ell \sqrt{4\pi(2\ell+1)}\, Y_\ell^0(\theta)$$

$$\times \frac{1}{2ikr}\left[e^{-ikr + i\ell\pi/2} - e^{ikr} e^{-i\ell\pi/2} e^{2i\delta_\ell} \right]. \quad (1)$$

We now allow the phase shift δ_ℓ to be complex, so that

$$e^{2i\delta_\ell} \equiv \eta_\ell = 1 + 2i e^{i\delta_\ell} \sin\delta_\ell$$

can have $|\eta_\ell| < 1$. Now the scattering amplitude is

$$f_k(\theta) = \frac{1}{k} \sum_{\ell=0}^{\infty} \sqrt{4\pi(2\ell+1)}\, Y_\ell^0(\theta)\, \frac{\eta_\ell - 1}{2i} \quad (2)$$

in which $\frac{\eta_\ell - 1}{2i} = e^{i\delta_\ell} \sin\delta_\ell$, and

$$\frac{d\sigma_{el}}{d\Omega} = |f_k(\theta)|^2.$$

The total elastic x-section is

$$\sigma_{el} = \frac{\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) \, |1-\eta_\ell|^2 .$$

Note that the absorption of the ℓth wave is greatest when $\eta_\ell = 0$. Yet even if $\eta_\ell = 0$, the elastic x-section of the ℓth wave is

$$\sigma_{el}^{(\ell)} = \frac{\pi}{k^2} (2\ell+1) > 0 \, !$$

This purely quantum effect is called <u>shadow scattering</u> because it peaks in the forward direction.

The probability current is

$$\vec{J} = \mathrm{Re}\left[\psi_k^*(r) \frac{\hbar}{i\mu} \nabla \psi_k(r) \right]$$

and so the amount of probability that is absorbed is the integral over a big sphere

$$\Delta P = - \int J \cdot dS .$$

Only

$$J_r = \mathrm{Re}\left[\psi_k^*(r) \frac{\hbar}{i\mu} \frac{\partial}{\partial r} \psi_k(r) \right]$$ counts.

Terms in J_r with different ℓ's integrate to zero over the unit sphere since the $Y_\ell^0(\theta)$'s are orthonormal.

$$\frac{\partial}{\partial r} \psi_k(\vec{r}) \sim + \sum_{\ell=0}^{\infty} i^\ell \sqrt{4\pi(2\ell+1)}\, Y_\ell^0(\theta) \times \left(\frac{e^{-ikr} e^{i\ell\pi/2} + \eta_\ell e^{ikr} e^{-i\ell\pi/2}}{2r} \right)$$

So, dropping terms with $\ell' \neq \ell$, we get

$$J_r \sim -\mathrm{Re}\left\{ \sum_{\ell=0}^{\infty} \frac{\hbar}{i\mu} \frac{4\pi(2\ell+1)}{(-i)4r^2 k} |Y_\ell^0(\theta)|^2 \times \left(e^{ikr} e^{-i\ell\pi/2} - \eta_\ell^* e^{-ikr+i\ell\pi/2} \right) \times \left(e^{-ikr} e^{i\ell\pi/2} + \eta_\ell e^{ikr-i\ell\pi/2} \right) \right\}$$

$$= -\frac{\pi}{\mu k} \frac{\hbar}{r^2} \sum_{\ell=0}^{\infty} \left(1 - |\eta_\ell|^2\right) |Y_\ell^0(\theta)|^2 (2\ell+1)$$

Hence ΔP is

$$\Delta P = -\int J_r r^2 d\Omega = + \frac{\pi}{\mu k} \hbar \sum_{\ell=0}^{\infty} (1 - |\eta_\ell|^2)(2\ell+1).$$

The absorption x-section then is

$$\sigma_{abs} = \frac{\Delta P}{\hbar k/\mu}$$

$$= \frac{\pi}{k^2} \sum_{\ell=0}^{\infty} \left(1 - |\eta_\ell|^2\right)(2\ell+1)$$

$$\sigma_{abs}^{(\ell)} = \frac{\pi}{k^2} \left(1 - |\eta_\ell|^2\right)(2\ell+1).$$

$$= \sigma_{el}^{(\ell)} - |\eta_\ell|^2 \frac{\pi(2\ell+1)}{k^2}.$$

Note that $\sigma_{abs} = 0$ when $|\eta_\ell| = 1$ or when δ_ℓ is real.

The total x-section is

$$\sigma_{tot} = \sigma_{el} + \sigma_{abs}$$

$$= \frac{\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1)\left(|1-\eta_\ell|^2 + 1 - |\eta_\ell|^2\right)$$

$$= \frac{\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1)\left(2 - \eta_\ell - \eta_\ell^*\right)$$

$$= \frac{2\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1)\left(1 - \mathrm{Re}\,\eta_\ell\right)$$

Since $Y_\ell^0(0) = \sqrt{(2\ell+1)/(4\pi)}$, the imaginary part of the <u>forward</u> scattering amplitude (z) is

$$\mathrm{Im}\, f_k(0) = \frac{1}{k} \sum_{\ell=0}^{\infty} (2\ell+1)\, \frac{1}{2}\, \mathrm{Re}\,(1-\eta_\ell)$$

and so we arrive at the optical theorem when the scattering is not necessarily elastic

$$\sigma_{tot} = \sigma_{el} + \sigma_{abs} = \frac{4\pi}{k}\, \mathrm{Im}\, f_k(0) .$$

This theorem is of universal validity (as far as we know).