

Schrödinger's Equation

We have seen that H and \vec{p} generate displacements in time and space

$$i\hbar \frac{\partial}{\partial t} \langle \vec{x}, t | \psi \rangle = \langle \vec{x}, t | H | \psi \rangle$$

and

$$\frac{\hbar}{i} \vec{\nabla} \langle \vec{x}, t | \psi \rangle = \langle \vec{x}, t | \vec{p} | \psi \rangle.$$

In non-relativistic quantum mechanics, we subtract mc^2 from the true hamiltonian for a single particle and take the limit as $v/c \rightarrow 0$.

So for a single free particle

$$H = H_{\text{true}} - mc^2$$

$$= \sqrt{c^2 p^2 + m^2 c^4} - mc^2$$

$$= mc^2 \sqrt{1 + \frac{p^2}{m^2 c^2}} - mc^2$$

$$= \frac{p^2}{2m} + \text{terms we neglect.}$$

A single particle of mass m interacting with a potential $V(\vec{x})$ has

$$H = \frac{\vec{p}^2}{2m} + V(\vec{x}).$$

Schrödinger's equation then is

$$\begin{aligned}
 i\hbar \frac{\partial}{\partial t} \langle \vec{x}, t | \psi \rangle &= \langle \vec{x}, t | H | \psi \rangle \\
 &= \langle \vec{x}, t | \left[\frac{\vec{p}^2}{2m} + V(\vec{x}) \right] | \psi \rangle \\
 &= \left[\left(\frac{\hbar}{i} \vec{\nabla} \right)^2 \frac{1}{2m} + V(\vec{x}) \right] \langle \vec{x}, t | \psi \rangle.
 \end{aligned}$$

Δ is an abbreviation for $\vec{\nabla}^2$. So

$$i\hbar \frac{\partial}{\partial t} \langle \vec{x}, t | \psi \rangle = \left[-\frac{\hbar^2}{2m} \Delta + V(\vec{x}) \right] \langle \vec{x}, t | \psi \rangle.$$

If $|\psi\rangle$ is an e-vec of H , with e-val E , then

$$i\hbar \frac{\partial}{\partial t} \langle \vec{x}, t | \psi \rangle = \langle \vec{x}, t | H | \psi \rangle = E \langle \vec{x}, t | \psi \rangle$$

which we may integrate to

$$\langle \vec{x}, t | \psi \rangle = e^{-iEt/\hbar} \langle \vec{x} | \psi \rangle$$

often written as

$$\psi(\vec{x}, t) = e^{-iEt/\hbar} \psi(\vec{x}).$$

For two particles interacting through a potential $V(\vec{x}_1, \vec{x}_2)$ the hamiltonian is

$$H = \frac{\vec{p}_1^2}{2m_1} + \frac{\vec{p}_2^2}{2m_2} + V(\vec{x}_1, \vec{x}_2),$$

The momentum operator is

$$\vec{P} = \vec{p}_1 + \vec{p}_2.$$

Notice that

$$[H, \vec{P}] = [V(\vec{x}_1, \vec{x}_2), \vec{P}] = 0$$

because

$$\begin{aligned} [\vec{x}_1, \vec{x}_2, \vec{P}] &= [\vec{x}_1, \vec{x}_2, \vec{p}_1 + \vec{p}_2] \\ &= i\hbar - i\hbar = 0. \end{aligned}$$

When H is time independent, the time-evolution operator is

$$U(t) = e^{-iHt/\hbar} \quad \text{and} \quad |\psi(t)\rangle = U(t)|\psi\rangle$$

When H does involve time but $[H(t'), H(t'')] = 0$

Then

$$|\psi, t\rangle = \exp\left[-i \int_0^t dt' \frac{H(t')}{\hbar}\right] |\psi\rangle.$$

so $U(t) = \exp\left[-i \int_0^t dt' H(t')/\hbar\right]$, so

$$i\hbar \frac{\partial}{\partial t} \langle x | \psi, t \rangle = \langle x | H(t) | \psi, t \rangle.$$

When $[H(t'), H(t'')] \neq 0$, then we use Dyson's formula

$$U(t) = T \exp\left[-i \int_0^t dt' H(t')\right]$$

in which the H's are time ordered,

$$U(t) = e^{-i \int dt H(t)/\hbar} \dots e^{-i \int dt H(t)/\hbar} e^{-i \int dt H(0)/\hbar}$$

$\frac{t}{dt}$ factors with time increasing to the left.

$$\psi(\vec{x}, t) = \langle \vec{x}, t | \psi \rangle = \langle \vec{0}, 0 | e^{i(\vec{p} \cdot \vec{x} - Ht)/\hbar} | \psi \rangle$$

So
$$| \vec{x}, t \rangle = e^{-i(\vec{p} \cdot \vec{x} - Ht)/\hbar} | \vec{0}, 0 \rangle.$$

So
$$U(\vec{x}, t) = e^{-i(\vec{p} \cdot \vec{x} - Ht)/\hbar}$$

moves a state from $\vec{0}, 0$ to \vec{x}, t .

So
$$e^{-i(\vec{p} \cdot \vec{x} - Ht)/\hbar} | \psi \rangle$$

actually moves the state $|\psi\rangle$ by \vec{x}, t .

But
$$\langle \vec{0}, 0 | e^{-i(\vec{p} \cdot \vec{x} - Ht)/\hbar} | \psi \rangle = \langle -\vec{x}, -t | \psi \rangle,$$

while
$$\langle \vec{0}, 0 | e^{i(\vec{p} \cdot \vec{x} - Ht)/\hbar} | \psi \rangle = \langle \vec{x}, t | \psi \rangle = \psi(\vec{x}, t).$$

Most often in textbooks one sees

$$\langle \vec{x}, t | \psi \rangle = \langle \vec{x} | e^{-iHt/\hbar} | \psi \rangle = \langle \vec{x} | \psi, t \rangle$$

But $e^{-iHt/\hbar} | \psi \rangle$ is really $|\psi\rangle$ moved backwards in time.

Look at picture on next page.

$$\langle \vec{x} + \vec{a}, t + b | \psi \rangle = \langle \vec{x}, t | \psi \rangle e^{i(a \cdot p - b H) / \hbar}$$

state moved
by $(-\vec{a}, -b)$

