QED

We will use "natural units" in which $\hbar = c = 1$ and $\alpha = e^2/4\pi \simeq 1/137$. Here $e > 0$.

The hamiltonian is

$$H = H_{0M} + H_{0F} + V$$

where

$$H_{0M} = \int d^3p \; p^0 \left[b_r^\dagger(p)\, b_r(p) + c_r^\dagger(p) c_r(p) - 1 \right]$$

and

$$H_{0F} = \int d^3k \; k^0 \left[a_r^\dagger(k)\, a_r(k) + \tfrac{1}{2} \right]$$

while in the "interaction picture"

$$V(t) = \int d^3x \; ie\, \bar{\psi}(\vec{x},t)\, \gamma^\mu \psi(\vec{x},t)\, A_\mu(\vec{x},t)$$

$$+ \; V_C(t)$$

in which

$$V_C(t) = \frac{1}{2} \int d^3x\, d^3y \; \frac{j^0(\vec{x},t)\, j^0(\vec{y},t)}{4\pi |\vec{x} - \vec{y}|}$$

is the Coulomb part of the potential V. Here $\psi(\vec{x},t)$ is the field of the electron and $\vec{A}_\mu(\vec{x},t)$ is that of the photons.

The operators $b_r(\vec{p})$ and $c_r(\vec{p})$ annihilate respectively an electron (positron) of momentum $\vec{p}$ and polarization r. They satisfy the <u>anti</u>-commutation relations

$$[\, b_r(p), b_{r'}(p') \,]_+ = 0$$

$$[\, b_r(p), b^\dagger_{r'}(p') \,]_+ = \delta_{rr'}\, \delta(\vec{p}-\vec{p}\,')$$

with similar rules for the c's. The a's are the photon operators you already know; they obey

$$[\, a_r(k), a^\dagger_{r'}(k') \,]_- = \delta_{rr'}\, \delta(\vec{k}-\vec{k}')$$

The photon field $A_\mu(\vec{x},0)$ is the one you have seen in SI units. Here in natural units, it is

$$A_\mu(\vec{x},t) = \sum_r \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \Big[e^{ip\cdot x} \epsilon_\mu(p,r)\, a(\vec{p},r) + e^{-ip\cdot x} \epsilon^*_\mu(p,r)\, a^\dagger(\vec{p},r) \Big]$$

where $p\cdot x = \vec{p}\cdot\vec{x} - p^0 t$

and $\vec{p}\cdot\epsilon(\vec{p},r) = 0$ and $\epsilon^0(\vec{p},r) = 0$

and $\epsilon(p,r)\cdot\epsilon(p,r') = \delta_{r,r'}$. As usual, we have

$$\sum_r \epsilon^i(\vec{p},r)\, \epsilon^{*j}(\vec{p},r) = \delta_{ij} - \hat{p}_i \hat{p}_j .$$

The *only* effect of V_c is to allow us to use a nicer formula for the mean value in the vacuum of the time-ordered product of $A_\mu(x)$ with $A_\nu(y)$:

$$-i\Delta_{\mu\nu}(x-y) \equiv \langle 0| T\{A_\mu(x)A_\nu(y)\}|0\rangle$$

$$= \int \frac{d^4q}{(2\pi)^4} \frac{-i\eta_{\mu\nu}}{q^2 - i\epsilon} e^{iq\cdot(x-y)}$$

where

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This function is "the propagator."

Dirac's γ-matrices satisfy the *anti*-commutation relation

$$[\gamma^\mu, \gamma^\nu]_+ = 2\eta^{\mu\nu}.$$

In 3+1 dimensions of space-time, they are four 4×4 matrices. One nice choice is

$$\gamma^0 = -i\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \text{ and } \vec{\gamma} = -i\begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

where $\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. $\beta \equiv \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$ and $\bar{\psi} \equiv \psi^\dagger \beta$.

The electron-positron field $\psi(x,t)$ is

$$\psi_\ell(x) = \sum_r \int \frac{d^3p}{(2\pi)^{3/2}} \left[u_\ell(\vec{p},r) e^{ip\cdot x} b(\vec{p},r) + v_\ell(\vec{p},r) e^{-ip\cdot x} c^\dagger(p,r) \right].$$

The u's and v's are called Dirac spinors. Details about them are explained in "Learning about Spin-One-Half Fields." Among them, the spin sums

$$\sum_r u_\ell(p,r)\, \overline{u}_{\ell'}(p,r) = \frac{1}{2p^0}\left(-i\gamma^\mu p_\mu + m\right)_{\ell\ell'}$$

and

$$\sum_r v_\ell(p,r)\, \overline{v}_{\ell'}(p,r) = \frac{1}{2p^0}\left(-i\gamma^\mu p_\mu - m\right)$$

are most useful.

The electron-positron propagator is

$$-i\Delta_{\ell m}(x,y) \equiv \langle 0 | \mathcal{T}\left(\psi_\ell(x)\, \psi^\dagger_m(y)\right) | 0 \rangle$$

$$\equiv \theta(x^0 - y^0) \langle 0 | \psi_\ell(x)\, \psi^\dagger_m(y) | 0 \rangle - \theta(y^0 - x^0) \langle 0 | \psi^\dagger_m(y)\, \psi_\ell(x) | 0 \rangle$$

$$= \int \frac{d^4q}{(2\pi)^4} \, -i \, \frac{\left[(-i\gamma^\mu q_\mu + m)\beta\right]_{\ell m}}{q^2 + m^2 - i\epsilon} e^{iq(x-y)} .$$

Note the extra minus sign!

A more useful form is

$$\langle 0 | \mathcal{T} \left(\psi_\ell(x) \bar{\psi}_m(y) \right) | 0 \rangle = \int \frac{d^4 q}{(2\pi)^4} \frac{-i(-i\gamma^\mu q_\mu + m)_{\ell m}}{q^2 + m^2 - i\epsilon} e^{iq(x-y)} .$$

The S-operator for QED is

$$S(\infty, -\infty) = \mathcal{T} \exp\left[-i \int d^4x \left(V - V_c \right) \right]$$

$$= \mathcal{T} \exp\left[e \int d^4x \, \bar{\psi}(x) \gamma^\mu A_\mu(x) \psi(x) \right] .$$