QED

We will use "natural units" in which \( \hbar = c = 1 \) and \( \alpha = e^2 / 4 \pi = 1 / 137 \). Here \( e > 0 \).

The hamiltonian is

\[
H = H_{\text{OM}} + H_{\text{OF}} + V
\]

where

\[
H_{\text{OM}} = \int d^3 p \ p_0 \left[ \begin{array}{c} b^+_r(p) b_r(p) + c^+_r(p) c_r(p) - 1 \end{array} \right]
\]

and

\[
H_{\text{OF}} = \int d^3 k \ k^0 \left[ a^+_r(k) a_r(k) + \frac{1}{2} \right]
\]

while in the "interaction picture"

\[
V(v) = \int d^3 x \ i e \ \bar{\Psi}(x', t) \gamma^0 \Psi(x', t) A^v(x', t)
\]

+ \( V_c(t) \)

in which

\[
V_c(t) = \frac{1}{2} \int d^3 x d^3 y \ j^0(r', t) j^0(r', t) \frac{1}{4 \pi |x - y|}
\]

is the Coulomb part of the potential \( V \).

Here \( \Psi(x', t) \) is the field of the electron and \( \bar{\Psi}(x', t) \) is that of the photon.
The operators \( b_\nu(p') \) and \( c_\nu(p') \) annihilate respectively an electron (positron) of momentum \( p' \) and polarization \( \nu \). They satisfy the anti-commutation relations

\[
\left[ b_\nu(p), b_{\nu'}(p') \right]_+ = 0
\]

\[
\left[ b_\nu(p), b_{\nu'}(p') \right]_- = \delta_{\nu\nu'} \delta(p - p')
\]

with similar rules for the \( c' \)'s. The \( a' \)'s are the photon operators you already know; they obey

\[
\left[ a_{\nu}(x'), a_{\nu'}(x) \right] = \delta_{\nu\nu'} \delta(x - x')
\]

The photon field \( A_\mu(x',0) \) is the one you have seen in SI units. Here in natural units, it is

\[
A_\mu(x',t) = \sum_n \frac{\delta(p_x)}{\sqrt{(2\pi)^3}} \left[ e^{i(p_x x - \omega p^0)} e^{i\epsilon(p,n) a(p,n) + \dagger} + e^{i\epsilon^*(p,n) a(p,n)^+} \right]
\]

where \( p \cdot x = p_x^2 + x^2 - p^0 t \)

and \( \epsilon_{\nu}(p',v) = 0 \) and \( \epsilon^0(p',v) = 0 \)

and \( \epsilon(p,v) \cdot \epsilon(p,v') = \delta_{v,v'} \). As usual, we have

\[
\sum_n \epsilon_i(p',v) \epsilon_j(p,v) = \delta_{ij} - \hat{p}_i \hat{p}_j .
\]
The only effect of $V_\circ$ is to allow us to use a nicer formula for the mean value in the vacuum of the time-ordered product of $A_\mu(x)$ with $A_\nu(y)$:

$$-i \Delta_{\mu\nu}(x-y) \equiv \langle 0 | \int \frac{d^4q}{(2\pi)^4} \frac{-i \eta_{\mu\nu}}{q^2 - i\epsilon} \delta(q \cdot (x-y)) | 0 \rangle$$

where

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This function is "the propagator".

Dirac's $\gamma$-matrices satisfy the anti-commutation relation

$$[\gamma^\mu, \gamma^\nu] = 2\gamma^{\mu\nu}.$$

In 3+1 dimensions of space-time, they are four 4x4 matrices. One nice choice is

$$\gamma^0 = -i \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad \text{and} \quad \gamma^2 = -i \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix},$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\sigma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $\sigma^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. $\beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ and $\bar{\gamma} = \gamma^+ \beta.$
The electron–positron field \( \psi(x,t) \) is
\[
\psi_e(x) = \sum \int \frac{d^3 p}{(2\pi)^3/2} \left[ \psi_e(p) e^{i p \cdot x} + \psi_e(p) e^{-i p \cdot x} \right].
\]

The \( \psi \)'s and \( \overline{\psi} \)'s are called Dirac spinors. Details about them are explained in "Learning about Spin–One–Half Fields." Among them, the spin sums
\[
\sum \frac{\psi_e(p,r)}{r} \overline{\psi_e}(p,r) = \frac{1}{2p^0} \left(-i \gamma^\mu p_\mu + m \right)_{00},
\]
and
\[
\sum \frac{\overline{\psi_e}(p,r)}{r} \overline{\psi_e}(p,r) = \frac{1}{2p^0} \left(-i \gamma^\mu p_\mu - m \right)
\]
are most useful.
The electron–positron propagator is
\[
-\delta_{e\mu} (x,y) \equiv \langle 0 | \mathcal{J} \left( \psi_e(x) \psi_e^\dagger (y) \right) | 0 \rangle
\]
\[
\equiv \Theta(x^0 - y^0) \langle 0 | \psi_e(x) \psi_e^\dagger (y) | 0 \rangle - \Theta(y^0 - x^0) \langle 0 | \psi_e^\dagger (y) \psi_e(x) | 0 \rangle
\]
\[
= \int \frac{d^4 \theta}{(2\pi)^4} -i \left[ (-i \gamma^\mu \theta_\mu + m) \delta^4 \theta \right] \frac{e^{i \theta \cdot (x-y)}}{\theta^2 + m^2 - i \varepsilon}.
\]

Note the extra minus sign!
A more useful form is:

\[<0 | \psi_k(x) \Phi_m(y) | 0> = \int \frac{d^4q}{(2\pi)^4} \frac{e^{-i(q \cdot k + m \cdot x)} \psi_k(x) \Phi_m(y)}{q^2 + m^2 - i\epsilon} \]

The S-operator for QED is:

\[S(\infty, -\infty) = \int \exp \left[ -i \int d^4x (V - V_0) \right] \]

\[= \int \exp \left[ e \int d^4x \nabla^\mu A^\mu(x) \nabla^\nu \Phi(x) \right] \]