Lie Algebra of Rotation Group

Group theory is hopeless, but the algebra of the generators of a group is fairly understandable. The most important of these Lie algebras is that of the 3 generators of rotations J_i:

$$[J_i, J_j] = i\hbar \sum_k \epsilon_{ijk} J_k. \qquad (1)$$

We have seen that $\vec{J}^2$ is a scalar

$$[J_i, \vec{J}^2] = 0. \qquad (2)$$

So we may simultaneously diagonalize any one of the 3 J_i's and $\vec{J}^2$. By convention, we choose J_3 and $\vec{J}^2$.

Define

$$J_+ = J_1 + iJ_2 \qquad (3)$$

$$J_- = J_1 - iJ_2. \qquad (4)$$

Note that

$$[J_3, J_+] = [J_3, J_1 + iJ_2] = i\hbar\left(\epsilon_{312} J_2 + i\epsilon_{321} J_1\right)$$

$$= i\hbar(J_2 - iJ_1) = \hbar(J_1 + iJ_2) = \hbar J_+. \qquad (5)$$

Also

$$[J_3, J_-] = [J_3, J_1 - iJ_2]$$

$$= i\hbar\left(\epsilon_{312} J_2 - i\,\epsilon_{321} J_1\right)$$

$$= \hbar\left(-J_1 + iJ_2\right) = -\hbar\left(J_1 - iJ_2\right)$$

$$= -\hbar J_- . \qquad (6)$$

And

$$[J_+, J_-] = [J_1 + iJ_2, J_1 - iJ_2]$$

$$= -i[J_1, J_2] + i[J_2, J_1]$$

$$= -2i[J_1, J_2] = (-2i)\, i\hbar\, \epsilon_{123} J_3$$

$$= 2\hbar J_3 . \qquad (7)$$

Similarly

$$[\vec{J}^2, J_+] = [\vec{J}^2, J_-] = [\vec{J}^2, J_3] = 0 . \quad (8)$$

Now

$$J_+ J_- = (J_1 + iJ_2)(J_1 - iJ_2)$$
$$= J_1^2 + J_2^2 - i[J_1, J_2]$$
$$= J_1^2 + J_2^2 - i(i\hbar)J_3$$
$$= J_1^2 + J_2^2 + \hbar J_3 \qquad (9)$$

Also

$$J_- J_+ = (J_1 - iJ_2)(J_1 + iJ_2)$$
$$= J_1^2 + J_2^2 + i[J_1, J_2]$$
$$= J_1^2 + J_2^2 + i(i\hbar) J_3$$
$$= J_1^2 + J_2^2 - \hbar J_3. \qquad (10)$$

Equivalently

$$J_+ J_- = \vec{J}^2 - J_3^2 + \hbar J_3 \qquad (11)$$

$$J_- J_+ = \vec{J}^2 - J_3^2 - \hbar J_3. \qquad (12)$$

So $\frac{1}{2}(J_+ J_- + J_- J_+) = \vec{J}^2 - J_3^2$ or

$$\vec{J}^2 = \frac{1}{2}(J_+ J_- + J_- J_+) + J_3^2. \qquad (13)$$

Let $|\psi\rangle$ be any normalized eigenstate of $\vec{J}^2$ so that $\vec{J}^2|\psi\rangle = \lambda|\psi\rangle$. Then

$$\lambda = \langle\psi|\vec{J}\cdot\vec{J}|\psi\rangle = \sum_{k=1}^{3} \langle\psi|J_k^2|\psi\rangle$$
$$= \sum_{k=1}^{3} \| J_k|\psi\rangle\|^2 \geq 0. \qquad (14)$$

So the e-vals of $\vec{J}^2$ must be non-negative. We set them to

$$j(j+1)\hbar^2 \quad \text{for} \quad j \geq 0. \qquad (15)$$

Note that if

$$j(j+1)\hbar^2 = \lambda \geq 0 \tag{16}$$

then

$$j(j+1) = \alpha = \lambda/\hbar^2 \geq 0 \tag{17}$$

has roots

$$j^2 + j - \alpha = 0$$

$$j = \frac{-1 \pm \sqrt{1+4\alpha}}{2}$$

only one of which

$$j = \frac{\sqrt{1+4\alpha} - 1}{2} \tag{18}$$

is nonnegative.

We will write the evals of J_3 as

$$m\hbar. \tag{19}$$

What are j and m? We must solve

$$\vec{J}^2 \, |j, m\rangle = j(j+1)\hbar^2 \, |j, m\rangle \tag{20}$$

and

$$J_3 \, |j, m\rangle = m\hbar \, |j, m\rangle. \tag{21}$$

Let's first see why

$$-j \le m \le j \,. \qquad (22)$$

Surely

$$0 \le \| J_+ |j\,m\rangle \|^2 = \langle j\,m | J_- J_+ | j\,m\rangle$$

$$= \langle j\,m | \vec{J}^2 - J_3^2 - \hbar J_3 | j\,m\rangle$$

$$= j(j+1)\hbar^2 - (m\hbar)^2 - m\hbar^2$$

$$= \hbar^2 \left[j(j+1) - m(m+1) \right]$$

$$= \hbar^2 \, (j-m)(j+m+1) \geq 0 \qquad (23)$$

as well as

$$0 \le \| J_- |j\,m\rangle \|^2 = \langle j\,m | J_+ J_- | j\,m\rangle$$

$$= \langle j\,m | \vec{J}^2 - J_3^2 + \hbar J_3 | j\,m\rangle$$

$$= \hbar^2 \left[j(j+1) - m(m-1) \right]$$

$$= \hbar^2 \, (j-m+1)(j+m) \geq 0 \,. \qquad (24)$$

Well

$$(j-m)(j+m+1) \geq 0 \quad \Rightarrow \quad -(j+1) \le m \le j \qquad (25)$$

and

$$(j-m+1)(j+m) \geq 0 \quad \Rightarrow \quad -j \le m \le j+1 \qquad (26)$$

Thus

$$-j \le m \le j \,. \qquad (27)$$

Next we want to see that if

$$J^2 |jm\rangle = \hbar^2 j(j+1) |jm\rangle \tag{28}$$

$$J_3 |jm\rangle = \hbar m |jm\rangle \tag{29}$$

then for $m = -j$

$$J_- |j, -j\rangle = 0 \tag{30}$$

and if $m > -j$

$$J^2 J_- |jm\rangle = \hbar^2 j(j+1) J_- |jm\rangle \tag{31}$$

and

$$J_3 J_- |jm\rangle = (m-1)\hbar J_- |jm\rangle, \tag{32}$$

Our result (24)

$$\| J_- |jm\rangle \|^2 = \hbar^2 (j-m+1)(j+m) \tag{33}$$

shows that $J_- |j, -j\rangle = 0$ (34)

which is (30). (Incidentally, since $j \geq 0$

$$0 = (j-m+1)(j+m) = j(j+1) - m(m-1) \tag{35}$$

implies $m = -j$ or $m = j+1$ which by (27) is impossible. So if $J_- |jm\rangle = 0$, then $m = -j$. (36))

Now assume $m > -j$. By (8)

$$[J^2, J_-] = 0 \tag{37}$$

and so

$$[J^2, J_-]\,|j\,m\rangle = 0 \tag{38}$$

or

$$\begin{aligned} J^2 J_- |j\,m\rangle &= J_- J^2 |j\,m\rangle \\ &= J_- \hbar^2 j(j+1)\,|j\,m\rangle \\ &= \hbar^2 j(j+1)\, J_- |j\,m\rangle \end{aligned} \tag{39}$$

which is (31). So $J_-|j\,m\rangle$ is an eigenstate of $\vec{J}^{\,2}$ with the same eigenvalue $\hbar^2 j(j+1)$ as the state $|j\,m\rangle$.

We continue to assume $m > -j$.

By (6)

$$[J_3, J_-] = -\hbar J_- \tag{40}$$

so

$$[J_3, J_-]\,|j\,m\rangle = -\hbar J_- |j\,m\rangle \tag{41}$$

or

$$\begin{aligned} J_3 J_- |j\,m\rangle &= J_- J_3 |j\,m\rangle - \hbar J_- |j\,m\rangle \\ &= J_- \hbar m\,|j\,m\rangle - \hbar J_- |j\,m\rangle \\ &= \hbar(m-1)\, J_- |j\,m\rangle \end{aligned} \tag{42}$$

which is (32). So if $m > -j$, then $J_-|j\,m\rangle$ is an e-vec of both J^2 and J_3 with e-vals $\hbar^2 j(j+1)$ and $\hbar(m-1)$.

What about $J_+|j\,m\rangle$?

Assume again that $|j\,m\rangle$ is an evec of J^2 and J_3 as in (28-29). Then we want to see that if $m=j$

$$J_+\,|j,j\rangle = 0 \tag{43}$$

and if $m<j$

$$J^2 J_+\,|j\,m\rangle = \hbar^2 j(j+1)\, J_+\,|j\,m\rangle \tag{44}$$

and

$$J_3 J_+\,|j\,m\rangle = \hbar\,(m+1)\, J_+\,|j\,m\rangle. \tag{45}$$

To prove (43), we use (23)

$$\|J_+\,|j\,m\rangle\|^2 = \hbar^2 (j-m)(j+m+1) \tag{46}$$

which says that $J_+\,|j\,j\rangle = 0$. (It's also true that $J_+\,|j\,m\rangle = 0$ implies $m=j$.)

To prove (44), we use (8)

$$0 = [J^2, J_+]\,|j\,m\rangle \tag{47}$$

so

$$J^2 J_+\,|j\,m\rangle = J_+ J^2\,|j\,m\rangle$$
$$= J_+\,\hbar^2 j(j+1)\,|j\,m\rangle \tag{48}$$

or

$$J^2 J_+\,|j\,m\rangle = \hbar^2 j(j+1)\, J_+\,|j\,m\rangle, \tag{49}$$

To prove (45), we use (5)

$$[J_3, J_+] = \hbar J_+ \tag{50}$$

to show that

$$[J_3, J_+]|jm\rangle = \hbar J_+ |jm\rangle \tag{51}$$

$$J_3 J_+ |jm\rangle = J_+ J_3 |jm\rangle + \hbar J_+ |jm\rangle$$

$$= \hbar m J_+ |jm\rangle + \hbar J_+ |jm\rangle$$

$$= \hbar(m+1) J_+ |jm\rangle \tag{52}$$

which is (45).

Now we find what j and m can be. Again we assume

$$\vec{J}^2 |jm\rangle = \hbar^2 j(j+1) |jm\rangle \tag{53}$$

$$J_3 |jm\rangle = \hbar m |jm\rangle . \tag{54}$$

By (27)

$$-j \le m \le j . \tag{55}$$

Let's write

$$m = j - x \tag{56}$$

and

$$m = -j + y . \tag{57}$$

What can x and y be?

By (55) both x and y must be non-negative

$$x \geq 0 \qquad (58)$$

$$y \geq 0. \qquad (59)$$

But if x were not an integer, then by repeatedly using (52), we could raise the value of m beyond j. And if y were not an integer, by using (42) over and over, we could lower m below $-j$.

For instance, if $x = \frac{1}{2}$, then by applying (52) once we'd have

$$J_3 \, J_+ |j, j - \tfrac{1}{2}\rangle = \hbar (j + \tfrac{1}{2}) J_+ |j, j - \tfrac{1}{2}\rangle \qquad (60)$$

which would violate (55).

So both x and y must be non-negative <u>integers</u>! That is,

$$j = m + k \qquad (61)$$

$$j = -m + l \qquad (62)$$

with $x = k$ and $y = l$, where k & l are integers.

The sum of (61) & (62) is

$$2j = k + l \qquad (63)$$

So $2j$ is a non-negative integer,

$$2j = 0, 1, 2, 3, 4 \ldots \qquad (64)$$

which means that

$$j = \frac{\text{non-negative integer}}{2}. \qquad (65)$$

So the possible values of j are $0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3$ etc.

By (42&52), the values of m run from $-j$ to j in steps of 1. So for a given j, there are $2j+1$ possible values of m. A given system will have one state $|j\,m\rangle$ for each m in the interval $-j \le m \le j$, $m-j$ integral.

We have seen (52) that

$$J_+ |j,m\rangle = c\,|j,m+1\rangle \tag{66}$$

as long as $m < j$. The constant c is restricted by the J_-J_+ relation (23)

$$\begin{aligned}\langle jm|J_-J_+|j'm\rangle &= |c|^2 \langle j\,m+1|j\,m+1\rangle \\ &= |c|^2 \\ &= \hbar^2 [j(j+1) - m(m+1)]\end{aligned} \tag{67}$$

since these states are normalized to unity. The conventional choice is

$$c = \hbar\sqrt{j(j+1) - m(m+1)} \tag{68}$$

so that

$$\begin{aligned}J_+|jm\rangle &= \hbar\sqrt{j(j+1)-m(m+1)}\;|j,m+1\rangle \\ &= \hbar\sqrt{(j-m)(j+m+1)}\;|j,m+1\rangle.\end{aligned} \tag{69}$$

Because the states are normalized, we have

$$\langle j,m+1|J_+|jm\rangle = \hbar\sqrt{(j-m)(j+m+1)} \tag{70}$$

whose complex conjugate is

$$\langle jm|J_-|j,m+1\rangle = \hbar\sqrt{(j-m)(j+m+1)} \quad \text{or}$$

$$\langle j,m-1|J_-|j,m\rangle = \hbar\sqrt{(j-m+1)(j+m)}\,. \tag{71}$$

And since $J_-|jm\rangle$ is an e-vec of J_3 with e-val $(m-1)\hbar$, it is orthonal to all e-vecs with different e-vals. Thus

$$J_-|jm\rangle = \hbar\sqrt{(j-m+1)(j+m)}\;|j,m-1\rangle. \quad (72)$$