

Ehrenfest's Theorem

Suppose

$$H = \frac{\vec{p}^2}{2m} + V(\vec{x}).$$

The H-picture $\vec{x}(t)$ is

$$\vec{x}(t) = e^{iHt/\hbar} \vec{x}(0) e^{-iHt/\hbar}$$

and so

$$\dot{x}_i = e^{iHt/\hbar} \frac{i}{\hbar} [H, x_i(0)] e^{-iHt/\hbar}$$

Note that

$$H(t) = e^{iHt/\hbar} H(0) e^{-iHt/\hbar} = H(0).$$

So

$$\dot{x}_i = e^{iHt/\hbar} \frac{i}{\hbar} \left[\frac{p^2}{2m}, x_i \right] e^{-iHt/\hbar}$$

$$[p^2, x_i] = [p_i^2, x_i] = p_i [p_i, x_i] + [p_i, x_i] p_i$$

$$= -i\hbar 2p_i = -2i\hbar p_i \quad \text{so}$$

$$\dot{x}_i = e^{iHt/\hbar} \frac{p_i(0)}{m} e^{-iHt/\hbar} = \frac{p_i(t)}{m}$$

Thus ..

$$x_i = e^{iHt/\hbar} \left[\frac{p_i(0)}{m} \right] e^{-iHt/\hbar}$$

So

$$m \ddot{x}_i(t) = e^{\frac{iHt}{\hbar}} \frac{i}{\hbar} [V(\vec{x}), p_i(0)] e^{-iHt/\hbar}$$

Problem 1.29 a, done in class, shows that

$$[V(\vec{x}), p_i] = i\hbar \frac{\partial V}{\partial x_i}$$

So we have

$$\begin{aligned} m \ddot{x}_i(t) &= e^{\frac{iHt}{\hbar}} \frac{i}{\hbar} i\hbar \frac{\partial V}{\partial x_i} e^{-iHt/\hbar} \\ &= - e^{iHt/\hbar} \frac{\partial V}{\partial x_i} e^{-iHt/\hbar} \\ &= - \frac{\partial V}{\partial x_i} (x(t)). \end{aligned}$$

So

$$m \ddot{\vec{x}}(t) = \dot{\vec{p}}(t) = - \vec{\nabla} V(\vec{x}(t)).$$

P. Ehrenfest 1927.

Clearly, if $|\psi\rangle$ is any H -pic state

$$\langle \psi | m \ddot{\vec{x}} | \psi \rangle = - \langle \psi | \vec{\nabla} V(\vec{x}(t)) | \psi \rangle.$$