Time-Dependent Perturbation Theory

We are going to have to write the hamiltonian $H = H_0 + V$ divided by $\hbar$ so often that it will help to use the abbreviations

$$h \equiv \frac{H}{\hbar}, \qquad h_0 \equiv \frac{H_0}{\hbar}, \quad \text{and} \quad v \equiv \frac{V}{\hbar}. \tag{1}$$

The time-translation operator then is

$$U(T) = e^{-ihT} = e^{-i(h_0+v)t}. \tag{2}$$

We will write this in a wacky way—as the product of a large number N of time slices of very brief duration $t = T/N$:

$$U(T) = \prod_{k=1}^{N} \left(e^{-i(h_0+v)t} \right). \tag{3}$$

Now we will cheat a bit (in a way that can be justified as $N \to \infty$)

$$U(T) = \prod_{k=1}^{N} \left(e^{-ih_0 t} e^{-ivt} \right) = \left[\prod_{k=1}^{N} \left(e^{-ih_0 t} e^{-ivt} \right) \right] e^{-ih_0 t}. \tag{4}$$

We rewrite this product in an even sillier way now starting from its right end: First

$$e^{-ih_0 t} e^{-ivt} e^{-ih_0 t} = e^{-2ih_0 t} e^{ih_0 t} e^{-ivt} e^{-ih_0 t} \tag{5}$$

and then

$$\begin{aligned} e^{-ih_0 t} e^{-ivt} e^{-ih_0 t} e^{-ivt} e^{-ih_0 t} &= e^{-ih_0 t} e^{-ivt} e^{-2ih_0 t} e^{ih_0 t} e^{-ivt} e^{-ih_0 t} \\ &= e^{-3ih_0 t} e^{2ih_0 t} e^{-ivt} e^{-2ih_0 t} e^{ih_0 t} e^{-ivt} e^{-ih_0 t}. \end{aligned} \tag{6}$$

As we continue, the operator $U(T)$ becomes the **k-ordered** product

$$\begin{aligned} U(T) &= e^{-i(N+1)ih_0 t} e^{Nih_0 t} e^{-ivt} e^{-Nih_0 t} \ldots e^{ih_0 t} e^{-ivt} e^{-ih_0 t} \\ &= e^{-i(N+1)ih_0 t} \mathcal{K} \left\{ \prod_{k=1}^{N} \left(e^{kih_0 t} e^{-ivt} e^{-kih_0 t} \right) \right\} \end{aligned} \tag{7}$$

The operator $\exp(-kih_0 t)$ is unitary. It translates the operator v to the time t in the **Heisenberg picture** of the free hamiltonian H_0. The operator

$$v_I(kt) \equiv e^{kih_0 t} v e^{-kih_0 t} \tag{8}$$

has the time dependence of the free hamiltonian H_0 and is said to be in the **interaction picture** denoted by the subscript I. It follows that

$$e^{-iv_I(kt)t} = e^{kih_0t}e^{-ivt}e^{-kih_0t}. \tag{9}$$

So we see that the time-evolution operator $U(T)$ is the **time ordered** product of exponentials of $-iv_I(kt)t$:

$$U(T) = e^{-i(N+1)ih_0t}e^{-iv_I(Nt)t}\dots e^{-iv_I(t)t} = e^{-i(N+1)ih_0t}\mathcal{T}\left\{\prod_{k=1}^{N} e^{-iv_I(kt)t}\right\}. \tag{10}$$

We now write the tiny duration $t = T/N$ as dt, so that the sum of the exponents is the time integral

$$-i\left[v_I(Ndt) + \cdots + v_I(dt)\right]dt = -i\int_0^T v_I(t)\,dt \tag{11}$$

and so we arrive at the formula

$$U(T) = e^{-ih_0T}\,\mathcal{T}\left\{e^{-i\int_0^T v_I(t)\,dt}\right\} = e^{-ih_0T}\,\mathcal{T}\left\{\exp\left[-i\int_0^T v_I(t)\,dt\right]\right\}. \tag{12}$$

Finally, with $v_I(t) = V_I(t)/\hbar$ and $h_0 = H_0/\hbar$, we have

$$U(T) = e^{-iH_0T/\hbar}\,\mathcal{T}\left\{e^{-i/\hbar\int_0^T V_I(t)\,dt}\right\} = e^{-iH_0T/\hbar}\,\mathcal{T}\left\{\exp\left[-\frac{i}{\hbar}\int_0^T V_I(t)\,dt\right]\right\} \tag{13}$$

in which

$$V_I(t) = \exp(itH_0/\hbar)V\exp(-itH_0/\hbar). \tag{14}$$

One may further show that the operator that translates from $t = -T$ to $t = T$ is

$$U(T,-T) = e^{-iH_0T/\hbar}\,\mathcal{T}\left\{\exp\left[-(i/\hbar)\int_{-T}^T V_I(t)\,dt\right]\right\}\,e^{-iH_0T/\hbar}. \tag{15}$$

The **S-operator** is this $U(T,-T)$ without the pre- and post-factors:

$$S(T,-T) = e^{iH_0T/\hbar}\,U(T,-T)\,e^{-iH_0T/\hbar} = \mathcal{T}\left\{\exp\left[-(i/\hbar)\int_{-T}^T V_I(t)\,dt\right]\right\}. \tag{16}$$

Between eigenstates of H_0 the operators S and U have the same matrix elements apart from a phase factor. Such matrix elements of the S-operator form the **S-matrix**.

In scattering problems, one often lets $T \to \infty$ so that

$$S \equiv S(\infty, -\infty) = \mathcal{T}\left\{\exp\left[-(i/\hbar)\int_{-\infty}^{\infty} V_I(t)\, dt\right]\right\}. \tag{17}$$

This operator is less scary when we expand the exponential:

$$\begin{aligned} S &= \mathcal{T}\left\{\exp\left[(-i/\hbar)\int_{-\infty}^{\infty} V_I(t)\, dt\right]\right\} \qquad (18) \\ &= 1 + \frac{-i}{\hbar}\int_{-\infty}^{\infty} V_I(t)\, dt + \frac{1}{2!}\left(\frac{-i}{\hbar}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{T}\left\{V_I(t_1)\, V_I(t_2)\right\} dt_1 dt_2 + \ldots \end{aligned}$$

An equivalent expression is

$$S = 1 - \frac{i}{\hbar}\int_{-\infty}^{\infty} V_I(t)\, dt - \frac{1}{\hbar^2}\int_{-\infty}^{\infty}\int_{-\infty}^{t_1} V_I(t_1)\, V_I(t_2)\, dt_1 dt_2 + \ldots \tag{19}$$

which is Dyson's expansion of the S-operator.