Bohr Frequencies

Suppose \( H \) has \( N \) eigenstates \( \lvert \text{E}_n \rangle \)

\[
H \lvert \text{E}_n \rangle = \text{E}_n \lvert \text{E}_n \rangle
\]

all different. Then \( I = \sum_{n=1}^{N} \text{E}_n \times \text{E}_n! \).

Also

\[
\frac{-iHt}{\hbar} \lvert \text{E}_n \rangle = E_n \lvert \text{E}_n \rangle = \text{E}_n \lvert \text{E}_n \rangle
\]

and so the mean value of any operator \( A \) in the state \( \lvert \text{E}_n \rangle \) will vary as

\[
\langle \text{E}_n \lvert U(t) \ A \ U(t) \lvert \text{E}_n \rangle = E_n \langle \text{E}_n \lvert A \lvert \text{E}_n \rangle \text{e}^{rac{-iE_n t}{\hbar}} \text{e}^{rac{iE_n t}{\hbar}}
\]

\[
= \langle \text{E}_n \lvert A \lvert \text{E}_n \rangle \text{e}^{rac{iE_n t}{\hbar}}
\]

which is to say, not at all. It is constant in time. That's why eigenstates of \( H \) are called stationary states.

An arbitrary state \( \lvert 14 \rangle \) can be expanded as

\[
\lvert 14 \rangle = I \lvert 14 \rangle = \sum_{n=1}^{N} \text{E}_n \times \lvert \text{E}_n \rangle \implies \lvert 14 \rangle \text{e}^{\frac{iE_n t}{\hbar}}
\]

So the mean value of any operator \( A \) will in the state \( \lvert 14 \rangle \) will evolve as

\[
\langle 14 \lvert U^*(t) \ A \ U(t) \lvert 14 \rangle = \sum_{n=1}^{N} \sum_{m=1}^{N} \langle \text{E}_m \lvert \text{E}_n \rangle \langle \text{E}_n \lvert A \lvert \text{E}_m \rangle \text{e}^{i(E_m - E_n) t/\hbar}.
\]
The frequencies

\[ W_{mn} = \frac{(E_m - E_n)}{\hbar} \]

are the Bohr frequencies. The time dependence of every operator is a linear combination of terms

\[ i (E_m - E_n) \frac{t}{\hbar} = i W_{mn} t \]

with coefficients \( \langle 4 | E_n | E_m | A | E_n \rangle \).

If \( A \) is compatible with \( H \), that is, if

\[ \{A, H\} = 0 \quad \text{(comp)} \]

then \( A \) and \( H \) are simultaneously diagonalizable. Equivalently, Eq. (comp) implies

\[ 0 = \langle E_m | [A, H] | E_n \rangle = (E_n - E_m) \langle E_m | A | E_n \rangle \]

Thus the mean value of any operator that commutes with \( H \) in an arbitrary state is

\[ \langle \{4, U(t) A U(t)^\dagger \} \rangle = \sum_{m=1}^{N} \langle 4 | E_n | E_m | A | E_n \rangle \langle E_n | A | E_n \rangle \]

only involves the diagonal matrix elements \( \langle E_n | A | E_n \rangle \) and so has no time dependence, due to \( U(t) \). Of course, if \( A \) explicitly depends on \( \theta \), then so does \( \langle \{4, U(t) A U(t)^\dagger \} \rangle \).