The Two-Body Problem

The hamiltonian for two non-relativistic particles of masses m_1 and m_2 interacting thru a potential

$$V = V(\vec{r}_1 - \vec{r}_2) \qquad (1)$$

is

$$H = \frac{\vec{p}_1^{\,2}}{2m_1} + \frac{\vec{p}_2^{\,2}}{2m_2} + V(\vec{r}_1 - \vec{r}_2). \qquad (2)$$

The total momentum $\vec{P}$ is

$$\vec{P} = \vec{p}_1 + \vec{p}_2. \qquad (3)$$

It generates translations and commutes with H

$$[\vec{P}, H] = 0. \qquad (4)$$

The center of mass R is

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \qquad (5)$$

and the separation $\vec{r}$ is

$$\vec{r} = \vec{r}_1 - \vec{r}_2. \qquad (6)$$

The relative momentum $\vec{p}$ is

$$\vec{p} = \frac{m_2 \vec{p}_1 - m_1 \vec{p}_2}{m_1 + m_2} . \qquad (7)$$

Its funny form ensures that it is canonically conjugate to the separation $\vec{r}$

$$[r_i, p_j] = [r_{1i} - r_{2i}, m_2 p_{1j} - m_1 p_{2j}] / (m_1 + m_2)$$

$$= (i\hbar\, m_2 + i\hbar\, m_1)\, \delta_{ij} / (m_1 + m_2)$$

$$= i\hbar\, \delta_{ij} \qquad (8)$$

and that it commutes with the center of mass $\vec{R}$

$$[R_i, p_j] = [m_1 r_{1i} + m_2 r_{2i}, m_2 p_{1j} - m_1 p_{2j}] / (m_1 + m_2)^2$$

$$= i\hbar\, \delta_{ij} (m_1 m_2 - m_1 m_2) / (m_1 + m_2)^2$$

$$= 0. \qquad (9)$$

The total momentum $\vec{P}$ and the center of mass $\vec{R}$ also are canonically conjugate

$$[R_i, P_j] = [m_1 r_{1i} + m_2 r_{2i}, p_{1j} + p_{2j}] / (m_1 + m_2)$$

$$= i\hbar\, \delta_{ij}. \qquad (10)$$

Finally, the separation $\vec{r}$ commutes with the total momentum $\vec{P}$

$$[r_i, P_j] = [r_{1i} - r_{2i}, p_{1j} + p_{2j}]$$
$$= i\hbar\,\delta_{ij}\,(1-1) = 0 \qquad (11)$$

which is why $[H, \vec{P}] = 0$ as in (4).

One may show that these definitions imply that the energy operator H is

$$H = \frac{\vec{P}^2}{2M} + \frac{\vec{p}^2}{2\mu} + V(\vec{r}) \qquad (12)$$

in which
$$M = m_1 + m_2 \qquad (13)$$
is the total mass, and

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \qquad (14)$$

is the reduced mass. For two particles of equal mass $m_1 = m_2 = m$, the reduced mass μ is $m/2$

$$\mu = \frac{m^2}{2m} = \frac{m}{2}. \qquad (15)$$

Since the center of mass $\vec{R}$ does not appear in H, it follows from (11) and from

$$[P_i, p_j] = 0 \tag{16}$$

that

$$[\vec{P}_i, H] = 0 \tag{17}$$

which is (4) again. But

$$i\hbar\dot{R}_i = [R_i, H] = \frac{1}{2M}[R_i, \vec{P}^2] = i\hbar\frac{P_i}{M} \tag{18}$$

and so the time evolution of the center of mass is

$$R_i(t) = R_i(0) + \frac{P_i(0)}{M}t \tag{19}$$

in which $P_i(t) = P_i(0)$ in view of (17).

The total angular momentum $\vec{L}$ is

$$\vec{L} = \vec{L}_1 + \vec{L}_2 = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 \tag{20}$$

$$= \vec{R} \times \vec{P} + \vec{r} \times \vec{p} \tag{21}$$

as you may show as a homework problem.