# Spin Is Angular Momentum

$$H = -eA^0 - c\gamma^0\left[(\vec{p} + \tfrac{e}{c}\vec{A})\cdot\vec{\gamma} - imc\right] \tag{1}$$

is Dirac's hamiltonian. Let's look at the case $\vec{A} = 0$ and $A^0 = A^0(r)$, that is, a spherically symmetric central potential.

Now

$$L_1 = x_2 p_3 - x_3 p_2 \tag{2}$$

and so

$$i\hbar \dot{L}_1 = [L_1, H] = [x_2 p_3 - x_3 p_2, -c\gamma^0 \vec{p}\cdot\vec{\gamma}] \tag{3}$$

$$= -c\gamma^0 i\hbar(\gamma_2 p_3 - \gamma_3 p_2) \neq 0. \tag{4}$$

Thus orbital angular momentum is not conserved. But

$$i\hbar \frac{\hbar}{2}\dot{\sigma}_{41} = \frac{\hbar}{2}[\sigma_{41}, H] = \frac{\hbar}{2}[\sigma_{41}, -c\gamma^0 \vec{p}\cdot\vec{\gamma}] \tag{5}$$

$$= -\frac{c\hbar}{2}[\sigma_{41}, \gamma^0 \vec{p}\cdot\vec{\gamma}]. \tag{6}$$

Now $\gamma^0 = -i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ and $\vec{\gamma} = -i\begin{pmatrix}0 & \vec{\sigma}\\ -\vec{\sigma} & 0\end{pmatrix}$ (7)

so

$$\gamma^0\vec{\gamma} = -\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & \vec{\sigma}\\ -\vec{\sigma} & 0\end{pmatrix} = -\begin{pmatrix}-\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix} = \vec{\Sigma} \tag{8}$$

where

$$\vec{\Sigma} = \begin{pmatrix}\vec{\sigma} & 0\\ 0 & -\vec{\sigma}\end{pmatrix}. \tag{9}$$

So

$$i\hbar \, \frac{\hbar}{2}\dot{\sigma}_{41} = -\frac{c\hbar}{2} p_i \left[\sigma_{41}, \Sigma_i\right] \qquad (10)$$

summed over $i$ from 1 to 3.

$$[\sigma_{41}, \Sigma_2] = \left[\begin{pmatrix}\sigma_1 & 0 \\ 0 & \sigma_1\end{pmatrix}, \begin{pmatrix}\sigma_2 & 0 \\ 0 & -\sigma_2\end{pmatrix}\right] \qquad 11$$

$$= \begin{pmatrix}[\sigma_1, \sigma_2] & 0 \\ 0 & -[\sigma_1, \sigma_2]\end{pmatrix} \qquad 12$$

$$= 2i\,\epsilon_{123}\Sigma_3 = 2i\Sigma_3 \qquad 13$$

$$[\sigma_{41}, \Sigma_3] = 2i\,\epsilon_{132}\Sigma_2 = -2i\Sigma_2. \qquad 14$$

So

$$i\hbar\left(\frac{\hbar}{2}\dot{\sigma}_{41}\right) = -\frac{c\hbar}{2}\left(p_2\, 2i\Sigma_3 - p_3\, 2i\Sigma_2\right) \qquad 15$$

$$= +ic\hbar\left(\Sigma_2 p_3 - \Sigma_3 p_2\right) \qquad 16$$

or in view of (8)

$$i\hbar\left(\frac{\hbar}{2}\dot{\sigma}_{41}\right) = +ic\hbar\,\gamma^0\left(\gamma_2 p_3 - \gamma_3 p_2\right). \qquad 17$$

Thus $\vec{J} = \vec{L} + \frac{\hbar}{2}\vec{\Sigma}$ is conserved, so spin is part of the angular momentum of an electrons.