Translations.

$$\langle x+dx|\alpha\rangle = \left(1+dx\frac{d}{dx}\right)\langle x|\alpha\rangle$$

is the infinitesimal form. The finite case is

$$\langle x+a|\alpha\rangle = e^{a\frac{d}{dx}}\langle x|\alpha\rangle.$$

And in 3 dimensions

$$\langle \vec{x}+\vec{a}|\alpha\rangle = e^{\vec{a}\cdot\vec{\nabla}}\langle \vec{x}|\alpha\rangle.$$

We will introduce a unitary operator

$$U(a) = e^{\frac{i\vec{a}\cdot\vec{p}}{\hbar}}$$

that translates $|\alpha\rangle$ so that

$$\langle \vec{x}+\vec{a}|\alpha\rangle = e^{\vec{a}\cdot\vec{\nabla}}\langle x|\alpha\rangle$$

$$= \langle x|e^{\frac{i\vec{a}\cdot\vec{p}}{\hbar}}|\alpha\rangle,$$

Here $\vec{p}^{\dagger} = \vec{p}$, so U is unitary.

$$U^{\dagger} = \left(e^{ia\cdot p/\hbar}\right)^{\dagger} = e^{-ia\cdot p/\hbar} = U^{-1}.$$

One of the surprises of quantum mechanics is that $\vec{p}$ is the momentum operator. Let's see why.

First, let's go back to the case of tiny $\vec{a}$

$$(1 + a \cdot \nabla)\langle x | \alpha \rangle = \langle x | 1 + \frac{i a \cdot p}{\hbar} | \alpha \rangle$$

So

$$a \cdot \nabla \langle x | \alpha \rangle = \frac{i \vec{a}}{\hbar} \cdot \langle x | \vec{p} | \alpha \rangle$$

So

$$\frac{\hbar}{i} \vec{\nabla} \langle \vec{x} | \alpha \rangle = \langle \vec{x} | \vec{p} | \alpha \rangle$$

represents this mysterious hermitian operator $\vec{p}$.

Dirac's rule is that commutators are related to Poisson brackets

$$[A, B] = i\hbar \{A, B\}_{PB}$$

$$= i\hbar \left(\frac{\partial A}{\partial x_k} \frac{\partial B}{\partial p_k} - \frac{\partial A}{\partial p_k} \frac{\partial B}{\partial x_k} \right)$$

So

$$[x_i, p_j] = i\hbar \left(\frac{\partial x_i}{\partial x_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial x_i}{\partial p_k} \frac{\partial p_j}{\partial x_k} \right) = i\hbar \delta_{ij}.$$

The identification $p_j = \frac{\hbar}{i} \frac{\partial}{\partial x_j}$
certainly satisfies Dirac's rule

$$[x_i, p_j] = \frac{\hbar}{i} [x_i, \frac{\partial}{\partial x_j}] = \frac{\hbar}{i} (-\delta_{ij})$$

$$= i\hbar \delta_{ij} = i\hbar \{x_i, p_j\}_{PB}.$$

But one could then ask why Dirac was right.

Before showing that $\vec{p}$ is momentum, let's consider time translations

$$\langle \vec{x}, t+dt | \alpha \rangle = \left(1 + dt \frac{\partial}{\partial t}\right) \langle \vec{x}, t | \alpha \rangle.$$

Again we introduce a unitary operator

$$U(\tau) = e^{-i\frac{H\tau}{\hbar}}$$

where $H^\dagger = H$ is hermitian so that $U^\dagger U = 1$.
Then

$$\left(1 + dt \frac{\partial}{\partial t}\right) \langle x, t | \alpha \rangle = \langle x, t | e^{-i\frac{H dt}{\hbar}} | \alpha \rangle$$

$$= \langle x, t | 1 - i\frac{H dt}{\hbar} | \alpha \rangle.$$

So we have,

$$dt \frac{\partial}{\partial t} \langle x, t | \alpha \rangle = -i \frac{dt}{\hbar} \langle x t | H | \alpha \rangle$$

so that

$$i\hbar \frac{\partial}{\partial t} \langle x, t | \alpha \rangle = \langle x, t | H | \alpha \rangle .$$

This H, as you know, is the energy operator, which is called the hamiltonian so as to confuse the biologists.

The finite time translation is

$$\langle \vec{x}, t + \tau | \alpha \rangle = \langle \vec{x}, t | e^{-i \frac{H\tau}{\hbar}} | \alpha \rangle .$$

The finite case for space and time is

$$\langle \vec{x} + \vec{a}, t + \tau | \alpha \rangle = \langle \vec{x}, t | e^{i(p \cdot a - H\tau)/\hbar} | \alpha \rangle$$

in which I took for granted that the hermitian operators $\vec{p}$ and H commute. (As fundamental operators, they do commute $[H, \vec{p}] = 0$, but when $\vec{p}$ is just the relative momentum, as in most bound-state problems, they often don't.)

When they don't commute, we can write

$$\langle x+a, t+\tau | \alpha\rangle = \langle \vec{x}, t | e^{ip\cdot a/\hbar} e^{-iH\tau/\hbar} |\alpha\rangle .$$

Suppose $|\alpha\rangle$ is an e-vec of both $\vec{p}$ and H when $[H, \vec{p}] = 0$. Then

$$\langle x+a, t+\tau | \alpha\rangle = \langle x, t | e^{i(p\cdot a - H\tau)/\hbar} |\alpha\rangle$$

$$= \langle x, t | e^{i(\vec{p}'\cdot\vec{a} - E'\tau)/\hbar} |\alpha\rangle$$

$$= e^{i(\vec{p}'\cdot\vec{a} - E'\tau)/\hbar} \langle x, t|\alpha\rangle$$

in which

$$\vec{p}\, |\alpha\rangle = \vec{p}\,' |\alpha\rangle$$

$$H |\alpha\rangle = E' |\alpha\rangle$$

and we might write

$$|\alpha\rangle = |\vec{p}\,', E'\rangle .$$

One further step

$$\langle x+a, t+\tau | p', E'\rangle = e^{i[p'\cdot(\vec{x}+a) - E'(t+\tau)]/\hbar} \langle \vec{0}, 0 | \alpha\rangle$$

or more simply

$$\langle \vec{x}', t | p', E'\rangle = e^{i(p'\cdot x - E't)/\hbar} \langle \vec{0}, 0|\alpha\rangle .$$

So the wave-function of an e-vec of $\vec{p}, H$ is a plane wave.

Now bring in special relativity:

$\begin{pmatrix} \vec{p} \\ H/c \end{pmatrix}$ is a 4-vec as is $\begin{pmatrix} \vec{x} \\ ct \end{pmatrix}$.

So the phase of this wave is a Lorentz scalar if we identify $\vec{p}$ with momentum and H with energy. That is nice because we'd like to represent Lorentz transformations by unitary operators

$$U_L \, |p'E'\rangle = |p''E''\rangle$$

$$U_L \, |\vec{x}, t\rangle = |\vec{x}', t'\rangle$$

$$\langle \vec{x}', t' | p'', E''\rangle = \langle \vec{x}, t | U_L^{+} U_L | p'E'\rangle$$

$$= \langle \vec{x}, t | p'E'\rangle$$

$$= e^{i(p'\cdot x - E't)/\hbar} \langle \vec{0}, 0 | \alpha\rangle$$

$$= e^{i(p''\cdot x' - E''t')/\hbar} \langle \vec{0}, 0 | \alpha\rangle$$

because $x \cdot p' - tE'$ is a scalar under L. transfs.

Suppose now that $|\alpha\rangle$ is a superposition of $|\vec{p}', E'\rangle$ states.

$$|\alpha\rangle = \int d^3p' \, |\vec{p}', E'\rangle\langle\vec{p}', E'|\alpha\rangle$$

where $E' = E'(\vec{p}')$. Then

$$\langle\vec{x}, t|\alpha\rangle = \int d^3p' \, \langle\vec{x}, t|p', E'\rangle\langle p' E'|\alpha\rangle$$

$$= \int \frac{d^3p'}{(2\pi\hbar)^{3/2}} e^{i(p'\cdot x - E't)/\hbar} \langle p' E'|\alpha\rangle .$$

The funny factor $\frac{1}{h^{3/2}} = \frac{1}{(2\pi\hbar)^{3/2}}$ is there to make

$$\langle\vec{x}'|\vec{x}''\rangle = \delta^3(x' - x'')$$

$$= \int d^3p' \, \langle x'|p'\rangle\langle p'|x''\rangle$$

$$= \int \frac{d^3p'}{(2\pi\hbar)^3} e^{ip'\cdot(x' - x'')/\hbar}$$

$$= \int \frac{d^3k}{(2\pi)^3} e^{ik\cdot(x' - x'')} = \delta^3(\overrightarrow{x' - x''}).$$

Let's go to 1-dimension and assume that $\langle p'E'|\alpha\rangle$ is a smooth function of $\vec{p}'$. Then the amplitude will be biggest when the phase is stationary

$$0 = \frac{d}{dp'}\left(p'x - E(p')t\right)\Big|_{p'=p_0}$$

$$0 = x - \frac{dE(p')}{dp'}t\Big|_{p_0}$$

or

$$v = \frac{dE(p')}{dp'}\Big|_{p_0}$$

If $E(p') = p'^2/(2m)$, then

$v = \frac{p_0}{m}$ makes sense as the group velocity. Here p_0 is the maximum of the smooth (real, positive) function $\langle p'E'|\alpha\rangle$.

In 3-D, with relativity, the group velocity is

$$\vec{v} = \nabla_p E(p) = \nabla_p \sqrt{c^4m^2 + \vec{p}^{\,2}c^2}$$

$$= \frac{\vec{p}_0 c^2}{\sqrt{c^4m^2 + p_0^2c^2}} = \frac{\vec{p}_0 c^2}{E_0} = \vec{v}_0.$$

in which we used

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1 - v^2/c^2}} \qquad E = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$$

so

$$\frac{\vec{p}c^2}{E} = \frac{\vec{v}}{c^2}c^2 = \vec{v}.$$