4. If $w(E)$ has a Lorentzian shape

$$w(E) = \frac{\Gamma\hbar}{2\pi}\frac{1}{(E-E_0)^2+\hbar^2\Gamma^2/4},$$

show that

$$c(t) = e^{-iE_0t/\hbar}\,e^{-\Gamma t/2}$$

and that the decay law is an exponential. The width of $w(E)$ is $\hbar\Gamma$, but ΔH is infinite, Thus ΔH is a rather poor measure of energy spread, and the width $\hbar\Gamma = \Delta E$ is the physically relevant quantity.

4.4.6 The solar neutrino puzzle

The nuclear reactions occurring in the interior of the Sun produce an abundance of electron neutrinos ν_e; 95% of these are produced in the reaction

$$p + p \to {}^2H + e^+ + \nu_e.$$

The Earth receives 6.5×10^{14} neutrinos per second and per square metre from the Sun. For about thirty years several experiments sought to detect these neutrinos, but all of them concluded that the measured neutrino flux is only about half the flux calculated using the standard solar model. Now this model is considered to be quite reliable,[17] in particular owing to recent results from helioseismology. In any case, the uncertainties in the solar model cannot explain this "solar neutrino deficit." The combined results of three experiments (see Footnote 4, Chapter 1) have now shown with no possible doubt that this neutrino deficit is due to the transformation of ν_e neutrinos into other types of neutrino during the passage from the Sun to the Earth. These experiments show that the total neutrino flux predicted by the solar model is correct, but that the measured *electron neutrino* flux is too small. We shall construct a simplified theory which gives the essential physics. We assume that

- there exist only two types of neutrino, the electron neutrino ν_e and the muon neutrino ν_μ (in fact, there is also a third type, the τ neutrino ν_τ);
- the entire phenomenon takes place in a vacuum during the propagation from the Sun to the Earth (the propagation inside the Sun actually plays an important role).[18]

It has long been thought that neutrinos have zero mass. If, on the contrary, they are massive, we can place them in their rest frame and write down the Hamiltonian in the $\{|\nu_e\rangle, |\nu_\mu\rangle\}$ basis:

$$|\nu_e\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \quad |\nu_\mu\rangle = \begin{pmatrix}0\\1\end{pmatrix}, \quad H = c^2\begin{pmatrix} m_e & m \\ m & m_\mu \end{pmatrix}.$$

[17] It is often said that the interior of the Sun is much better understood than that of the Earth.

[18] See E. Abers, *Quantum Mechanics*, New Jersey: Pearsons Education (2004), Chapter 6, for an elementary discussion.

The off-diagonal element m makes transitions between electron neutrinos and muon neutrinos possible.

1. Show that the states of definite mass are $|\nu_1\rangle$ and $|\nu_2\rangle$:

$$|\nu_1\rangle = \cos\frac{\theta}{2}\,|\nu_e\rangle + \sin\frac{\theta}{2}\,|\nu_\mu\rangle,$$
$$|\nu_2\rangle = -\sin\frac{\theta}{2}\,|\nu_e\rangle + \cos\frac{\theta}{2}\,|\nu_\mu\rangle,$$

with

$$\tan\theta = \frac{2m}{m_e - m_\mu},$$

and that the masses m_1 and m_2 are

$$m_1 = \frac{m_e + m_\mu}{2} + \sqrt{m^2 + \left(\frac{m_e - m_\mu}{2}\right)^2},$$
$$m_2 = \frac{m_e + m_\mu}{2} - \sqrt{m^2 + \left(\frac{m_e - m_\mu}{2}\right)^2}.$$

2. Neutrinos propagate with a speed close to that of light; their energy is very high compared with $\langle m\rangle c^2$, where $\langle m\rangle$ is the typical mass in H. Show that if an electron neutrino is produced inside the Sun at time $t = 0$ with state vector

$$|\varphi(t=0)\rangle = |\nu_e\rangle = \cos\frac{\theta}{2}\,|\nu_1\rangle - \sin\frac{\theta}{2}\,|\nu_2\rangle,$$

the state vector at time t has component on $|\nu_e\rangle$ given by

$$\langle\nu_e|\varphi(t)\rangle = e^{-iE_1 t/\hbar}\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\,e^{-i\Delta E\,t/\hbar}\right),$$

where $\Delta E = E_2 - E_1$. Show that the probability of finding a neutrino ν_e at time t is

$$\mathsf{p}_e(t) = 1 - \sin^2\theta\,\sin^2\left(\frac{\Delta E\,t}{2\hbar}\right).$$

This transformation phenomenon is called *neutrino oscillation.*

3. If $p \gg \langle m\rangle c$ is the neutrino momentum, show that ΔE, as measured in the Sun rest frame, is

$$\Delta E = \frac{(m_2^2 - m_1^2)c^3}{2p} = \frac{\Delta m^2\,c^3}{2p}$$

with $\Delta m^2 = m_2^2 - m_1^2$. Then t must also be measured in the Sun rest frame, and not in the neutrino rest frame!

4. Assuming that half an oscillation occurs during the trip from the Sun to the Earth (that is, $\Delta E\,t/\hbar = \pi$) for neutrinos of energy 8 MeV, what is the order of magnitude of the difference of the squared masses Δm^2? The Earth–Sun separation is 150 million kilometers.