Complement G_{VII}

EXERCISES

1. Particle in a cylindrically symmetrical potential

Let ρ, φ, z be the cylindrical coordinates of a spinless particle ($x = \rho \cos \varphi, y = \rho \sin \varphi ; \rho \geqslant 0, 0 \leqslant \varphi < 2\pi$). Assume that the potential energy of this particle depends only on ρ, and not on φ and z. Recall that:

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2}$$

a. Write, in cylindrical coordinates, the differential operator associated with the Hamiltonian. Show that H commutes with L_z and P_z. Show from this that the wave functions associated with the stationary states of the particle can be chosen in the form:

$$\varphi_{n,m,k}(\rho, \varphi, z) = f_{n,m}(\rho)\, e^{im\varphi}\, e^{ikz}$$

where the values that can be taken on by the indices m and k are to be specified.

b. Write, in cylindrical coordinates, the eigenvalue equation of the Hamiltonian H of the particle. Derive from it the differential equation which yields $f_{n,m}(\rho)$.

c. Let Σ_y be the operator whose action, in the $\{ | \mathbf{r} \rangle \}$ representation, is to change y to $-y$ (reflection with respect to the xOz plane). Does Σ_y commute with H? Show that Σ_y anticommutes with L_z, and show from this that $\Sigma_y | \varphi_{n,m,k} \rangle$ is an eigenvector of L_z. What is the corresponding eigenvalue? What can be concluded concerning the degeneracy of the energy levels of the particle? Could this result be predicted directly from the differential equation established in (*b*)?

2. Three-dimensional harmonic oscillator in a uniform magnetic field

N.B. The object of this exercise is to study a simple physical system for which the effect of a uniform magnetic field can be calculated exactly. Thus, it is possible in this case to compare precisely the relative importance of the "paramagnetic" and "diamagnetic" terms, and to study in detail the modification of the wave function of the ground state due to the effect of the diamagnetic term. (The reader may wish to refer to complements D_{VI} and B_{VII}.)

Consider a particle of mass μ, whose Hamiltonian is:

$$H_0 = \frac{\mathbf{P}^2}{2\mu} + \frac{1}{2}\mu\omega_0^2\,\mathbf{R}^2$$

(an isotropic three-dimensional harmonic oscillator), where ω_0 is a given positive constant.

a. Find the energy levels of the particle and their degrees of degeneracy. Is it possible to construct a basis of eigenstates common to H_0, $\mathbf{L}^2$, L_z?

b. Now, assume that the particle, which has a charge q, is placed in a uniform magnetic field $\mathbf{B}$ parallel to Oz. We set $\omega_L = -qB/2\mu$. The Hamiltonian H of the particle is then, if we choose the gauge $\mathbf{A} = -\frac{1}{2}\mathbf{r} \times \mathbf{B}$:

$$H = H_0 + H_1(\omega_L)$$

where H_1 is the sum of an operator which is linearly dependent on ω_L (the paramagnetic term) and an operator which is quadratically dependent on ω_L (the diamagnetic term). Show that the new stationary states of the system and their degrees of degeneracy can be determined exactly.

c. Show that if ω_L is much smaller than ω_0, the effect of the diamagnetic term is negligible compared to that of the paramagnetic term.

d. We now consider the first excited state of the oscillator, that is, the states whose energies approach $5\hbar\omega_0/2$ when $\omega_L \longrightarrow 0$. To first order in ω_L/ω_0, what are the energy levels in the presence of the field $\mathbf{B}$ and their degrees of degeneracy (the Zeeman effect for a three-dimensional harmonic oscillator)? Same questions for the second excited state.

e. Now consider the ground state. How does its energy vary as a function of ω_L (the diamagnetic effect on the ground state)? Calculate the magnetic susceptibility χ of this state. Is the ground state, in the presence of the field $\mathbf{B}$, an eigenvector of $\mathbf{L}^2$? of L_z? of L_x? Give the form of its wave function and the corresponding probability current. Show that the effect of the field $\mathbf{B}$ is to compress the wave function about Oz (in a ratio $[1 + (\omega_L/\omega_0)^2]^{1/4}$) and to induce a current.