Special Problem 7.4

The Dirac equation for a free spin-1/2 particle of mass m is

$$\sum_{b=1}^{4} \left(\hbar \sum_{\mu=0}^{3} \gamma^{\mu}_{ab} \frac{\partial}{\partial x^{\mu}} + mc\, \delta_{ab} \right) \psi_b(x) = 0 \tag{1}$$

in which $\psi(x)$ is a spinor with four components $\psi_b(x)$ for $b = 1 \ldots 4$. With a double use of summation conventions and in natural units with $\hbar = c = 1$, one may write this as

$$(\gamma^{\mu} \partial_{\mu} + m)\, \psi(x) = 0. \tag{2}$$

Consider any four linearly independent 4-spinors, for example,

$$\chi^{(c)}_b(x) = \delta_{bc}\, e^{ipx/\hbar} \tag{3}$$

with b, c running from 1 to 4 and

$$px = \sum_{\mu=0}^{3} p_{\mu} x^{\mu} = \mathbf{p} \cdot \mathbf{x} - p^0 x^0 = \mathbf{p} \cdot \mathbf{x} - (E/c)(ct) = \mathbf{p} \cdot \mathbf{x} - E\, t. \tag{4}$$

Show that the four 4-spinors $\psi_a(x)$ defined by

$$\psi^{(c)}_a(x) = \sum_{b=1}^{4} \left(\hbar \sum_{\mu=0}^{3} \gamma^{\mu}_{ab} \frac{\partial}{\partial x^{\mu}} - mc\, \delta_{ab} \right) \chi^{(c)}_b(x) \tag{5}$$

or more succinctly by

$$\psi^{(c)}(x) = (\gamma^{\mu} \partial_{\mu} - m)\, \chi^{(c)}(x) \tag{6}$$

are solutions of the free Dirac Equation (1) provided the 4-vector p satisfies a secret condition. What is that condition?

Hint: Use the condition

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta^{\mu\nu} \tag{7}$$

in which

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{8}$$