The Harmonic Oscillator

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}.$$

Note that the particle of mass m is attached to a spring of spring constant

$$-kx = -\frac{\partial H}{\partial x} = -m\omega^2 x$$

that is $k = m\omega^2$ or $\omega = \sqrt{k/m}$.
So translational invariance is gone; this p is not the full momentum.

What makes H easy to solve is that it is quadratic in x and p. In fact, with

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right) \qquad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right)$$

H is

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$

$$= \hbar\omega\left[\left(\frac{m\omega}{2\hbar}\right)\left(x - \frac{ip}{m\omega}\right)\left(x + \frac{ip}{m\omega}\right) + \frac{1}{2}\right]$$

$$= \hbar\omega\left[\left(\frac{m\omega}{2\hbar}\right)\left(x^2 + \frac{p^2}{m^2\omega^2} + \frac{i}{m\omega}[x,p]\right) + \frac{1}{2}\right]$$

$$H = \hbar\omega\left[\left(\frac{m\omega}{2\hbar}\right)\left(x^2 + \frac{p^2}{m^2\omega^2} - \frac{\hbar}{m\omega}\right) + \frac{1}{2}\right]$$

$$= \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2.$$

Note that

$$[a, a^\dagger] = \frac{m\omega}{2\hbar}\left[x + \frac{ip}{m\omega}, x - \frac{ip}{m\omega}\right]$$

$$= -\frac{i}{2\hbar}[x, p] + \frac{i}{2\hbar}[p, x]$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

So

$$[a, a^\dagger] = 1. \qquad \text{And } [a^\dagger, a] = -1.$$

Let's look for the ground state $|0\rangle$. Its energy E_0 is

$$E_0 = \langle 0|H|0\rangle = \hbar\omega \langle 0|a^\dagger a|0\rangle + \frac{\hbar\omega}{2}.$$

So $E_0 \geq \hbar\omega/2$, and to get $E_0 = \frac{\hbar\omega}{2}$ we must arrange that

$$a|0\rangle = 0.$$

We can solve $\langle x|a|0\rangle = 0.$

$$0 = \langle x|a|0\rangle = \sqrt{\frac{m\omega}{2\hbar}} \langle x \,|\, x + \frac{ip}{m\omega} \,|\, 0\rangle$$

So we need

$$x \langle x|0\rangle = -\frac{i}{m\omega}\langle x|p|0\rangle = -\frac{\hbar}{m\omega}\frac{d}{dx}\langle x|0\rangle.$$

the solution is obtained by writing

$$\frac{d \log \langle x|0\rangle}{dx} = \frac{\frac{d\langle x|0\rangle}{dx}}{\langle x|0\rangle} = -\frac{m\omega}{\hbar}x$$

so that

$$\log \langle x|0\rangle = -\frac{m\omega^2}{\hbar}\frac{x^2}{2} + c$$

or

$$\langle x|0\rangle = N e^{-\frac{m\omega^2}{2\hbar}x^2},$$

$$1 = N^2 \int_{-\infty}^{\infty} dx \, e^{-\frac{m\omega^2 x^2}{\hbar}} = N^2 \sqrt{\frac{\hbar}{m\omega^2}} \int_{-\infty}^{\infty} dy \, e^{-y^2}$$

$$= N^2 \sqrt{\frac{\pi\hbar}{m\omega^2}} \quad \text{so} \quad N = \left(\frac{m\omega^2}{\pi\hbar}\right)^{1/4}.$$

So

$$\langle x|0\rangle = \left(\frac{m\omega^2}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega^2}{2\hbar}x^2}.$$

So $$H|0\rangle = \frac{\hbar\omega}{2}|0\rangle.$$

Now we saw that $[a, a^\dagger] = 1$
and that

$$H = \hbar\omega\left(a^\dagger a + \tfrac{1}{2}\right).$$

So

$$[H, a^\dagger] = \hbar\omega\,[a^\dagger a, a^\dagger]$$

$$= \hbar\omega\left(a^\dagger a a^\dagger - a^\dagger a^\dagger a\right)$$

$$= \hbar\omega\, a^\dagger [a, a^\dagger] = \hbar\omega\, a^\dagger.$$

Thus if $\quad H|E\rangle = E|E\rangle$, then

$$H a^\dagger|E\rangle = \left([H, a^\dagger] + a^\dagger H\right)|E\rangle$$

$$= \left(\hbar\omega\, a^\dagger + E a^\dagger\right)|E\rangle$$

$$= (\hbar\omega + E)\, a^\dagger|E\rangle$$

That is, $a^\dagger|E\rangle$ is an e-vec of H
with e-val $\quad E + \hbar\omega$.

Thus $\quad H|0\rangle = \frac{\hbar\omega}{2}|0\rangle \quad$ implies that

$$H a^\dagger|0\rangle = \left(\frac{\hbar\omega}{2} + \hbar\omega\right)a^\dagger|0\rangle = \hbar\omega\left(1 + \tfrac{1}{2}\right)a^\dagger|0\rangle.$$

And so

$$H\, a^{+2}|0\rangle = \hbar\omega\left(2+\tfrac{1}{2}\right)a^{+2}|0\rangle.$$

In general

$$H\,(a^+)^n|0\rangle = \hbar\omega\left(n+\tfrac{1}{2}\right)(a^+)^n|0\rangle.$$

The e-vecs of H are

$$|n\rangle = C_n\,(a^+)^n|0\rangle$$

where C_n is a normalization constant.

The number operator is

$$N = a^+a$$

and so

$$H = \hbar\omega\left(a^+a+\tfrac{1}{2}\right) = \hbar\omega\left(N+\tfrac{1}{2}\right).$$

$$N|n\rangle = a^+a\,|n\rangle = \left(\frac{H}{\hbar\omega}-\frac{1}{2}\right)|n\rangle$$

$$= \left(\frac{\hbar\omega\left(n+\frac{1}{2}\right)}{\hbar\omega}-\frac{1}{2}\right)|n\rangle = n|n\rangle.$$

$$H = \hbar\omega\left(N+\tfrac{1}{2}\right).$$

$$a^+a|n\rangle = n|n\rangle.$$

$$N a|n\rangle = a^\dagger a\, a|n\rangle$$

$$= \left([a^\dagger a, a] + a\, a^\dagger a\right)|n\rangle$$

$$= \left(a^\dagger a a - a\, a^\dagger a + a n\right)|n\rangle$$

$$= \left([a^\dagger, a]a + a n\right)|n\rangle$$

$$= (n-1)\, a|n\rangle.$$

So

$$a|n\rangle = d_n\, |n-1\rangle,$$

where d_n is a normalization constant.

$$1 = |d_n|^2 \langle n-1|n-1\rangle = |d_n|^2$$

$$= \langle n|a^\dagger a|n\rangle = n\langle n|n\rangle = n$$

So $\quad d_n = \sqrt{n}$

apart from an irrelevant phase. That is,

$$a|n\rangle = \sqrt{n}\,|n-1\rangle.$$

Similarly,

$$N a^\dagger|n\rangle = a^\dagger a\, a^\dagger|n\rangle = \left([a^\dagger a, a^\dagger] + a^\dagger a^\dagger a\right)|n\rangle$$

$N a^\dagger |n\rangle = a^\dagger a a^\dagger |n\rangle =$

$$= \left(a^\dagger a a^\dagger - a^\dagger a^\dagger a + a^\dagger a^\dagger a \right) |n\rangle$$

$$= \left(a^\dagger [a, a^\dagger] + a^\dagger a^\dagger a \right) |n\rangle$$

$$= \left(a^\dagger + a^\dagger n \right) |n\rangle = (n+1)\, a^\dagger |n\rangle.$$

So

$$a^\dagger |n\rangle = e_n |n+1\rangle,$$

$$1 = |e_n|^2 \langle n+1 | n+1 \rangle = |e_n|^2$$

$$= \langle n | a a^\dagger | n \rangle = \langle n | a^\dagger a + 1 | n \rangle$$

$$= (n+1) \langle n | n \rangle = n+1.$$

So

$$e_n = \sqrt{n+1} \quad \text{and}$$

$$a^\dagger |n\rangle = \sqrt{n+1}\, |n+1\rangle$$

apart from an irrelevant phase.

The normalized states $|n\rangle$ are

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle .$$

For every complex number α, the operator

$$D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$$

is unitary since

$$D^\dagger(\alpha) D(\alpha) = \exp(\alpha^* a - \alpha a^\dagger)\exp(\alpha a^\dagger - \alpha^* a) = 1.$$

Let

$$a(\lambda) = D^\dagger(\lambda\alpha)\, a\, D(\lambda\alpha).$$

Then

$$a(0) = a \qquad \text{and}$$

$$\frac{d a(\lambda)}{d\lambda} = D^\dagger(\lambda\alpha)\,[a, \alpha a^\dagger - \alpha^* a]\, D(\lambda\alpha)$$

$$= \alpha\, D^\dagger(\lambda\alpha) D(\lambda\alpha) = \alpha.$$

So

$$a(1) = D^\dagger(\alpha)\, a\, D(\alpha) = a(0) + \int_0^1 d\lambda\, \alpha$$

or

$$D^\dagger(\alpha)\, a\, D(\alpha) = a + \alpha.$$

This is why Glauber calls $D(\alpha)$ a displacement operator.

Since $D(\alpha)$ is unitary, the coherent state

$$|\alpha\rangle = D(\alpha)|0\rangle \quad \text{is normalized.}$$

and is an e-vec of a with e-val α because

$$D^\dagger(\alpha)\, a\, |\alpha\rangle = D^\dagger(\alpha)\, a\, D(\alpha)|0\rangle$$

$$= (a+\alpha)|0\rangle = \alpha|0\rangle$$

since $a|0\rangle = 0$. Thus

$$a|\alpha\rangle = \alpha\, D(\alpha)|0\rangle = \alpha|\alpha\rangle.$$

This state $|\alpha\rangle = D(\alpha)|0\rangle$ is a coherent state.

Suppose A and B are two operators whose commutator $[A,B]$ commutes with both A and B

$$[A,[A,B]] = [B,[A,B]] = 0.$$

Let's compute

$$e^A e^B = \left(1 + A + \frac{A^2}{2} + \frac{A^3}{3!} + \dots\right)\left(1 + B + \frac{B^2}{2} + \frac{B^3}{3!}\right)$$

$$= 1 + A + B + \frac{1}{2}A^2 + AB + \frac{1}{2}B^2 + \frac{A^2 B}{2} + \frac{AB^2}{2} + \frac{A^3}{3!} + \frac{B^3}{3!} + \dots$$

So

$$e^{A} e^{B} = 1 + A + B + \tfrac{1}{2}(A+B)^2 + \tfrac{1}{2}[A,B]$$

$$+ \tfrac{1}{3!}(A+B)^3 + \tfrac{1}{4}[A,B]^2 + \cdots$$

$$= e^{A+B} e^{\frac{1}{2}[A,B]} .$$

So when A & B commute with $[A,B]$, we have

$$e^{A} e^{B} = e^{A+B} e^{\frac{1}{2}[A,B]} .$$

Application:

$$e^{\alpha a^{\dagger}} e^{-\alpha^{*} a} = D(\alpha)\, e^{-\frac{1}{2}|\alpha|^2 [a^{\dagger}, a]}$$

$$= e^{\frac{1}{2}|\alpha|^2} D(\alpha)$$

so

$$D(\alpha) = e^{\alpha a^{\dagger}} e^{-\alpha^{*} a} e^{-\frac{1}{2}|\alpha|^2}$$

and

$$|\alpha\rangle = D(\alpha)|0\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^{\dagger}} e^{-\alpha^{*} a} |0\rangle$$

$$= e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^{\dagger}} |0\rangle$$

$$= e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{(\alpha a^{\dagger})^n}{n!} |0\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle .$$

The inner products

$$\langle n|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \frac{\alpha^n}{\sqrt{n!}}$$

mean that the probabilities

$$P(n) = |\langle n|\alpha\rangle|^2$$
$$= \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2}$$

form a Poisson distribution.

Incidentally,

$$\langle\beta|\alpha\rangle = \exp(-\tfrac{1}{2}|\beta|^2 - \tfrac{1}{2}|\alpha|^2 + \beta^*\alpha)$$

$$|\langle\beta|\alpha\rangle|^2 = e^{-|\alpha-\beta|^2}$$

$$\mathbb{1} = \int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha|$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} d\,\mathrm{Re}\,\alpha \int d\,\mathrm{Im}\,\alpha \; |\alpha\rangle\langle\alpha|.$$