Addition of Angular Momenta

Suppose our system has two angular momenta $\vec{J}_1$ and $\vec{J}_2$. Then one set of states is the e-vecs of $\vec{J}_1^{\,2}$, $\vec{J}_2^{\,2}$, J_{1z}, J_{2z}, & $J_{1z}+J_{2z}$. These states are the direct products

$$|j_1, j_2, m_1, m_2\rangle = |j_1, m_1\rangle \otimes |j_2, m_2\rangle \qquad (1)$$

like the states $|n\,\ell\, m \pm \tfrac{1}{2}\rangle$ that diagonalized the important parts $H_0 + H_1$ of the H of an H-atom in a constant magnetic field. We have

$$J_1^2 |j_1 j_2 m_1 m_2\rangle = \left(J_1^2 |j_1 m_1\rangle\right) \otimes |j_2 m_2\rangle$$

$$= \hbar^2 j_1(j_1+1) |j_1 m_1\rangle \otimes |j_2 m_2\rangle$$

$$= \hbar^2 j_1(j_1+1)\, |j_1 j_2 m_1 m_2\rangle \qquad (2)$$

$$J_2^2 |j_1 j_2 m_1 m_2\rangle = |j_1 m_1\rangle \otimes J_2^2 |j_2 m_2\rangle$$

$$= \hbar^2 j_2(j_2+1)\, |j_1 j_2 m_1 m_2\rangle \qquad (3)$$

$$J_{1z} |j_1 j_2 m_1 m_2\rangle = \left(J_{1z} |j_1 m_1\rangle\right) \otimes |j_2 m_2\rangle$$
$$= \hbar m_1 |j_1 j_2 m_1 m_2\rangle \qquad (4)$$

$$J_{2z}\,|j_1 j_2 m_1 m_2\rangle = |j_1 m_1\rangle \otimes \left(J_{2z}\,|j_2 m_2\rangle\right)$$

$$= \hbar m_2\,|j_1 j_2 m_1 m_2\rangle \qquad (5)$$

$$(J_{1z}+J_{2z})\,|j_1 j_2 m_1 m_2\rangle = J_{1z}\,|j_1 j_2 m_1 m_2\rangle + J_{2z}\,|j_1 j_2 m_1 m_2\rangle$$

$$= \hbar(m_1+m_2)\,|j_1 j_2 m_1 m_2\rangle. \qquad (6)$$

These states $|j_1 j_2 m_1 m_2\rangle$ form a complete orthonormal set of states for the system of $\vec{J}_1$ and $\vec{J}_2$.

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 m_1' m_2'\rangle = \delta_{m_1 m_1'}\,\delta_{m_2 m_2'}. \qquad (7)$$

But we also could find e-vecs of $\vec{J}_1^{\,2}$, $\vec{J}_2^{\,2}$,

$$J_z = J_{1z} + J_{2z}, \qquad (8)$$

and

$$\vec{J}^{\,2} = (\vec{J}_1 + \vec{J}_2)^2. \qquad (9)$$

First, let's check that $\vec{J}$ is an angular momentum, that is, that it satisfies the rule

$$[J_i, J_j] = i\hbar \sum_{k=1}^{3} \epsilon_{ijk} J_k. \qquad (10)$$

Well, since the two J's are independent

$$[J_{1i}, J_{2j}] = 0 \qquad (11)$$

we have

$$[J_i, J_j] = [J_{1i} + J_{2i}, J_{1j} + J_{2j}]$$

$$= [J_{1i}, J_{1j}] + [J_{2i}, J_{2j}]$$

$$= i\hbar \sum_k \epsilon_{ijk} J_{1k} + i\hbar \sum_k \epsilon_{ijk} J_{2k}$$

$$= i\hbar \sum_k \epsilon_{ijk} (J_{1k} + J_{2k})$$

$$= i\hbar \sum_k \epsilon_{ijk} J_k , \qquad (12)$$

So $\vec{J} = \vec{J}_1 + \vec{J}_2$ is an angular momentum. Since it is, we know that

$$[J^2, J_z] = 0 \qquad (13)$$

which is just the statement that J^2 is a scalar.

Now note that by (11)

$$\vec{J}^2 = (\vec{J}_1 + \vec{J}_2)^2 = \vec{J}_1^{\,2} + J_2^2 + 2J_1 \cdot J_2 \qquad (14)$$

$$= \vec{J}_1^{\,2} + \vec{J}_2^{\,2} + 2 J_{1z} J_{2z} + 2 J_{1x} J_{2x} + 2 J_{1y} J_{2y}$$

$$= \vec{J}_1^{\,2} + \vec{J}_2^{\,2} + 2 J_{1z} J_{2z} + (J_{1x} + iJ_{1y})(J_{2x} - iJ_{2y})$$
$$+ (J_{1x} - iJ_{1y})(J_{2x} + iJ_{2y}) \qquad (15)$$

so

$$\vec{J}^2 = J_1^2 + J_2^2 + 2 J_{1z} J_{2z} + J_{1+} J_{2-} + J_{1-} J_{2+} . \qquad (16)$$

Thus $[\vec{J}^2, \vec{J}_1^2] = [\vec{J}^2, \vec{J}_2^2] = 0.$ (17)

So $\vec{J}^2$, J_z, $\vec{J}_1^2$, and $\vec{J}_2^2$ all commute with each other. We can find simultaneous e-vecs $|j_1 j_2 j m\rangle$

$$\vec{J}_1^2 \, |j_1 j_2 j m\rangle = \hbar^2 j_1(j_1+1) \, |j_1 j_2 j m\rangle$$

$$\vec{J}_2^2 \, |j_1 j_2 j m\rangle = \hbar^2 j_2(j_2+1) \, |j_1 j_2 j m\rangle$$

$$\vec{J}^2 \, |j_1 j_2 j m\rangle = \hbar^2 j(j+1) \, |j_1 j_2 j m\rangle$$

$$J_z \, |j_1 j_2 j m\rangle = \hbar m \, |j_1 j_2 j m\rangle \qquad (18)$$

in which

$$-j \le m \le j. \qquad (19)$$

Note, however, that $[\vec{J}^2, J_{1z}] \neq 0 \neq [\vec{J}^2, J_{2z}]$ because, e.g., by (16) and $[J_z, J_\pm] = \pm \hbar J_\pm$ (3.5.66)

$$[J^2, J_{1z}] = [J_{1+}, J_{1z}] J_{2-} + [J_{1-}, J_{1z}] J_{2+}$$

$$= -\hbar J_{1+} J_{2-} + \hbar J_{1-} J_{2+} \neq 0 . \qquad (20)$$

So we can't add J_{1z} or J_{2z} to J^2, J_z, J_1^2 & J_2^2.

So we have two complete orthonormal sets of states — the $(2j_1+1)(2j_2+1)$ states

$$|j_1 j_2 m_1 m_2\rangle \tag{21}$$

and the states

$$|j_1 j_2 j m\rangle. \tag{22}$$

Among the states (21) the one with the highest value of J_z is the state

$$|j_1 j_2\ m_1 = j_1\ m_2 = j_2\rangle = |j_1 j_2 j_1 j_2\rangle \tag{23}$$

$$J_z |j_1 j_2 j_1 j_2\rangle = (J_{1z} + J_{2z}) |j_1 j_2 j_1 j_2\rangle$$

$$= \hbar (j_1 + j_2) |j_1 j_2 j_1 j_2\rangle. \tag{24}$$

So some linear combination of the states (22) should have the same e-value of J_z

$$J_z |j_1 j_2\ j m\rangle = \hbar (j_1 + j_2) |j_1 j_2 j m\rangle$$

$$= \hbar m |j_1 j_2 j m\rangle. \tag{25}$$

and this should be the highest possible e-value of J_z. But the state $|j_1 j_2\ j m\rangle$ with the highest m is

$$|j_1 j_2\ j j\rangle \tag{26}$$

for the highest value of j. Thus the highest value of j must be

$$j = j_1 + j_2 . \tag{27}$$

What is the minimum value of j? Call it j_0. Then the states $|j_1 j_2 j m\rangle$ consist of

$$\begin{array}{ll} 2j_0+1 & |j_1 j_2 j_0 m\rangle\text{'s} \\ & \\ 2(j_0+1)+1 & |j_1 j_2 j_0+1\ m\rangle\text{'s} \end{array} \tag{28}$$

up to

$$2(j_1+j_2)+1 \qquad |j_1 j_2, j_1+j_2, m\rangle\text{'s}.$$

There must be as many $|j_1 j_2 j m\rangle$ states as $|j_1 j_2 m_1 m_2\rangle$ states. So

$$\sum_{j=j_0}^{j_1+j_2} 2j+1 = (2j_1+1)(2j_2+1)$$

$$= \sum_{j=0}^{j_1+j_2} 2j+1 - \sum_{j=0}^{j_0-1} 2j+1 . \tag{29}$$

Use $\sum_{n=0}^{N} n = \frac{N}{2}(N+1)$ and $\sum_{n=0}^{N} 1 = N+1$ (30)

to find $(2j_1+1)(2j_2+1) =$

$$2\frac{(j_1+j_2)}{2}(j_1+j_2+1) + j_1+j_2+1 - 2\frac{j_0}{2}(j_0-1) - j_0 . \tag{31}$$

So j_0 is a solution of

$$(2j_1+1)(2j_2+1) = (j_1+j_2)(j_1+j_2+1) + j_1+j_2+1 - j_0^2 \qquad (31)$$

or

$$j_0^2 = (j_1+j_2)(j_1+j_2+1) + j_1+j_2+1 - (2j_1+1)(2j_2+1)$$

$$= j_1^2+j_2^2+2j_1j_2+2j_1+2j_2+1 - 4j_1j_2 - 2j_1 - 2j_2 - 1$$

$$= (j_1-j_2)^2. \qquad (32)$$

That is, the minimum value of j must be

$$j = |j_1 - j_2|. \qquad (33)$$

So the states $|j_1 j_2\, j\, m\rangle$ run from

$$|j_1 j_2, |j_1-j_2|, m\rangle \text{ to } |j_1 j_2, j_1+j_2, m\rangle. \qquad (34)$$

Since

$$1 = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1 j_2\, m_1 m_2\rangle\langle j_1 j_2\, m_1 m_2| \qquad (35)$$

we have

$$|j_1 j_2\, j\, m\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1 j_2\, m_1 m_2\rangle\langle j_1 j_2\, m_1 m_2|j_1 j_2\, j\, m\rangle. \qquad (36)$$

The numbers

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 j m \rangle \qquad (37)$$

are the Clebsch-Gordan coefficients (CGc's).

They vanish unless $m_1 + m_2 = m$ since

$$0 = J_z - J_{1z} - J_{2z} \qquad (38)$$

whence

$$0 = \langle j_1 j_2 m_1 m_2 | \; J_z - J_{1z} - J_{2z} \; | j_1 j_2 j m \rangle$$

$$= \hbar (m - m_1 - m_2) \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m \rangle . \qquad (39)$$

So

$$m - m_1 - m_2 \neq 0 \quad \Longrightarrow \quad \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m \rangle = 0. \qquad (40)$$

We easily may find two recursion relations for the CGc's by using (3.5.39-40)

$$J_\pm | j m \rangle = \sqrt{(j \mp m)(j \pm m + 1)} \; \hbar \, | j, m \pm 1 \rangle \qquad (41)$$

We apply $J_\pm = J_{1\pm} + J_{2\pm}$ to Eq. (36):

$$J_\pm \, |j_1 j_2 j m\rangle = \hbar\sqrt{(j\mp m)(j\pm m+1)}\; |j_1 j_2 j\, m\pm 1\rangle$$

$$= (J_{1\pm} + J_{2\pm}) \sum_{m_1' m_2'} |j_1 j_2 m_1' m_2'\rangle\langle j_1 j_2 m_1' m_2' | j_1 j_2 j m\rangle$$

$$= \sum_{m_1' m_2'} \Big[\left(J_{1\pm}\, |j_1 m_1'\rangle\right)\otimes |j_2 m_2'\rangle + |j_1 m_1'\rangle \otimes \left(J_{2\pm}\, |j_2 m_2'\rangle\right)\Big]$$

$$\times \langle j_1 j_2 m_1' m_2' | j_1 j_2 j m\rangle$$

$$= \hbar \sum_{m_1 m_2} \Big[\sqrt{(j_1 \mp m_1')(j_1 \pm m_1' + 1)}\; |j_1 j_2\, m_1' \pm 1\, m_2'\rangle$$

$$+ \sqrt{(j_2 \mp m_2')(j_2 \pm m_2' + 1)}\; |j_1 j_2\, m_1\, m_2' \pm 1\rangle \Big]$$

$$\times \langle j_1 j_2 m_1' m_2' | j_1 j_2 j m\rangle \,. \qquad (42)$$

We now take the inner product of this ket equation with $|j_1 j_2 m_1 m_2\rangle$ using the ON rule

$$0 = \langle j_1 j_2 m_1 m_2 | j_1 j_2\, m_1' \pm 1\, m_2'\rangle$$

$$\text{unless} \quad m_1 = m_1' \pm 1 \quad \text{and} \quad m_2 = m_2' \qquad (43)$$

and

$$0 = \langle j_1 j_2 m_1 m_2 | j_1 j_2\, m_1'\, m_2' \pm 1\rangle \quad \text{unless}$$

$$m_1 = m_1' \quad \text{and} \quad m_2 = m_2' \pm 1 . \qquad (44)$$

Thus

$$\sqrt{(j \mp m)(j \pm m+1)} \quad \langle j_1 j_2 m_1 m_2 | j_1 j_2 j\, m \pm 1\rangle$$

$$= \sqrt{(j_1 \mp (m_1 \mp 1))(j_1 \pm (m_1 \mp 1)+1)} \quad \langle j_1 j_2\, m_1 \mp 1\, m_2 | j_1 j_2 j m\rangle$$

$$+ \sqrt{(j_2 \mp (m_2 \mp 1))(j_2 \pm (m_2 \mp 1)+1)} \quad \langle j_1 j_2, m_1, m_2 \mp 1 | j_1 j_2 j m\rangle \qquad (45)$$

$$= \sqrt{(j_1 \mp m_1 + 1)(j_1 \pm m_1)} \quad \langle j_1 j_2 \;\; m_1 \mp 1 \;\; m_2 | j_1 j_2 j m\rangle$$

$$+ \sqrt{(j_2 \mp m_2 + 1)(j_2 \pm m_2)} \quad \langle j_1, j_2, m_1, m_2 \mp 1 | j_1 j_2 j m\rangle \, . \qquad (46)$$

For the simple cases most often of interest, these relations are pretty easy to use. For instance, an H-atom in a magnetic field has e-vecs whose angular parts are

$$|l \tfrac{1}{2} m \pm \tfrac{1}{2}\rangle \qquad (47)$$

when we neglect the $\vec{A}^2$ term. We can combine these states to form e-vecs of J^2 and J_z. The top state with $j = l + \frac{1}{2}$ and $m = j$ is

$$|l, \tfrac{1}{2}, l+\tfrac{1}{2}, l+\tfrac{1}{2}\rangle = |l, \tfrac{1}{2}, l, \tfrac{1}{2}\rangle. \qquad (48)$$

On it J_- gives by (41)

$$J_-|l \tfrac{1}{2}\, l+\tfrac{1}{2}\, l+\tfrac{1}{2}\rangle = \sqrt{(l+\tfrac{1}{2}+l+\tfrac{1}{2})(l+\tfrac{1}{2}-l-\tfrac{1}{2}+1)}\, \hbar\, |l \tfrac{1}{2}\, l+\tfrac{1}{2}\, l-\tfrac{1}{2}\rangle$$

$$= \hbar\sqrt{2l+1} \;\; |l \tfrac{1}{2}\, l+\tfrac{1}{2}\, l-\tfrac{1}{2}\rangle \qquad (49)$$

$$= \left(J_{1-} + J_{2-}\right) | l \tfrac{1}{2}\ l \tfrac{1}{2} \rangle$$

$$= \left(L_- + S_-\right) | l \tfrac{1}{2}\ l \tfrac{1}{2} \rangle$$

$$= \left(L_- | l l \rangle\right) \otimes | \tfrac{1}{2} \tfrac{1}{2} \rangle + | l l \rangle \otimes \left(S_- | \tfrac{1}{2} \tfrac{1}{2} \rangle\right)$$

$$= \hbar \sqrt{2l}\, | l\ l-1 \rangle \otimes | \tfrac{1}{2} \tfrac{1}{2} \rangle + | l l \rangle \otimes \hbar \sqrt{1 \cdot 1}\, | \tfrac{1}{2} -\tfrac{1}{2} \rangle \qquad (42)$$

So

$$\sqrt{2l+1}\ | l \tfrac{1}{2}\ l+\tfrac{1}{2}\ l-\tfrac{1}{2} \rangle = \sqrt{2l}\ | l \tfrac{1}{2}\ l-1\ \tfrac{1}{2} \rangle + | l \tfrac{1}{2}\ l\ -\tfrac{1}{2} \rangle \qquad (43)$$

or

$$| l \tfrac{1}{2}, l+\tfrac{1}{2}\ l-\tfrac{1}{2} \rangle = \sqrt{\frac{2l}{2l+1}}\ | l \tfrac{1}{2}\ l-1\ \tfrac{1}{2} \rangle + \frac{1}{\sqrt{2l+1}} | l \tfrac{1}{2}\ l\ -\tfrac{1}{2} \rangle \qquad (44)$$

which is normalized.

The state $| l, \tfrac{1}{2}, l-\tfrac{1}{2}, l-\tfrac{1}{2} \rangle$ must be orthogonal to $| l, \tfrac{1}{2}, l+\tfrac{1}{2}, l-\tfrac{1}{2} \rangle$:

$$| l \tfrac{1}{2}\ l-\tfrac{1}{2}\ l-\tfrac{1}{2} \rangle = \frac{1}{\sqrt{2l+1}}\ | l \tfrac{1}{2}\ l-1\ \tfrac{1}{2} \rangle - \sqrt{\frac{2l}{2l+1}}\ | l, \tfrac{1}{2}, l, -\tfrac{1}{2} \rangle \qquad (45)$$

apart from an arbitrary over-all factor of ±1.

The next step is to apply

$$J_- = L_- + S_- \text{ to both } | l, \tfrac{1}{2}, l+\tfrac{1}{2}, l-\tfrac{1}{2} \rangle \ \& \ | l \tfrac{1}{2}\ l-\tfrac{1}{2}, l-\tfrac{1}{2} \rangle. \qquad (46)$$

This procedure will relate all the states

$$|l,\tfrac{1}{2},l+\tfrac{1}{2},m\rangle \quad \text{for} \quad -l-\tfrac{1}{2} \le m \le l+\tfrac{1}{2} \tag{47}$$

and

$$|l,\tfrac{1}{2},l-\tfrac{1}{2},m\rangle \quad \text{for} \quad -l+\tfrac{1}{2} \le m \le l-\tfrac{1}{2} \tag{48}$$

to the states $|l,\tfrac{1}{2},m,\pm\tfrac{1}{2}\rangle$ for $-l \le m \le l$. (49)

More generally, to add $\vec{J}_1+\vec{J}_2$, one starts with the top state

$$|j_1,j_2,\underset{\overset{\uparrow}{j}}{j_1+j_2},\underset{\overset{\uparrow}{m}}{j_1+j_2}\rangle = |j_1,j_2,\underset{\overset{\nearrow}{m_1}}{j_1},\underset{\overset{\nearrow}{m_2}}{j_2}\rangle \tag{50}$$

where

and applies $J_- = J_{1-}+J_{2-}$ using the rule

$$J_-|j,m\rangle = \hbar\sqrt{(j+m)(j-m+1)}\;|j,m-1\rangle. \tag{51}$$

We get

$$\begin{aligned}
J_-|j_1,j_2,j_1+j_2,j_1+j_2\rangle &= \hbar\sqrt{(2j_1+2j_2)}\;|j_1,j_2,j_1+j_2,j_1+j_2-1\rangle \\
&= \left(J_{1-}|j_1,j_1\rangle\right)\otimes|j_2,j_2\rangle + |j_1,j_1\rangle\otimes J_{2-}|j_2,j_2\rangle \\
&= \hbar\sqrt{2j_1}\;|j_1,j_1-1\rangle|j_2,j_2\rangle + |j_1,j_1\rangle\hbar\sqrt{2j_2}\;|j_2,j_2-1\rangle \\
&= \hbar\left(\sqrt{2j_1}\;|j_1,j_2,j_1-1,j_2\rangle + \sqrt{2j_2}\;|j_1,j_2,j_1,j_2-1\rangle\right)
\end{aligned} \tag{52}$$

So

$$|j_1 j_2, j_1+j_2, j_1+j_2-1\rangle = \sqrt{\frac{j_1}{j_1+j_2}}\,|j_1 j_2, j_1-1, j_2\rangle + \sqrt{\frac{j_2}{j_1+j_2}}\,|j_1 j_2 j_1, j_2-1\rangle . \quad (53)$$

The orthogonal state is

$$|j_1 j_2, j_1+j_2-1, j_1+j_2-1\rangle = \sqrt{\frac{j_2}{j_1+j_2}}\,|j_1 j_2, j_1-1, j_2\rangle$$

$$- \sqrt{\frac{j_1}{j_1+j_2}}\,|j_1 j_2\, j_1\, j_2-1\rangle \quad (54)$$

to within a phase factor conventionally chosen as ±1.

One continues applying J_- to successive $|j_1, j_2, jm\rangle$ states until one has expressed all $(2j_1+1)(2j_2+1)$ of them in terms of the $|j_1 j_2 m_1 m_2\rangle$ states.