Use of the Uncertainty Principle

The H for an electron in the electrostatic field of a (fixed) proton is

$$H = \frac{p^2}{2m} + V(r) = \frac{p^2}{2m} - \frac{e^2}{r}$$

in which $m = m_e$ and $e^2/\hbar c \approx 1/137$.
The electron gets as close or $\sim \Delta r$ to the proton as possible consistent with keeping down the kinetic energy $p^2/2m \sim (\Delta p)^2/2m$ where we guess

$$\Delta p \, \Delta r = \hbar \gtrsim \frac{\hbar}{2}.$$

So the ground-state energy E is

$$E = \frac{p^2}{2m} - \frac{e^2}{r} = \frac{p^2}{2m} - e^2 \frac{p}{\hbar}$$

in which we set $\Delta p = p$, $\Delta r = r$, $\Delta p \Delta r = \hbar$.
We minimize $E(p)$

$$0 = \frac{p}{m} - \frac{e^2}{\hbar} \Rightarrow p = \frac{me^2}{\hbar}.$$

So $r = \frac{\hbar}{p} = \frac{\hbar^2}{me^2}$ and

$$E = \left(\frac{me^2}{\hbar}\right)^2 \frac{1}{2m} - e^2 \frac{me^2}{\hbar^2} = -\frac{1}{2} mc^2 \left(\frac{e^2}{\hbar c}\right)^2.$$

In these units, the dimensionless ratio

$$\alpha = \frac{e^2}{\hbar c} \simeq \frac{1}{137.036}$$

is the fine-structure constant.
Our result then is

$$E = -\frac{1}{2}(mc^2)\left(\frac{1}{137.04}\right)^2$$

$$= -\frac{1}{2}\frac{(.511 \times 10^6)}{(137.04)^2} = -13.6\ eV$$

which is the exact answer. $E_n^Z = -\frac{1}{2}mc^2\frac{(Z\alpha)^2}{n^2}$.

This is luck. Had we set

$$\Delta p \Delta r = \frac{\hbar}{2}$$

we'd have gotten $-54.4\,eV$.

But in general the wise use of the uncertainty principle can get one to within a factor of 10 of the true ground-state energy.