Rotations

Under a right-handed rotation by the small angle θ about the axis $\vec{\theta}$ the vector $\vec{r}$ changes to

$$\vec{r} + \vec{\delta r} = \vec{r} + \vec{\theta} \times \vec{r}.$$

Here $\vec{\theta} \cdot \vec{\theta} = \theta^2$ and

$$(\theta \times r)_k = \sum_{i,j=1}^{3} \epsilon_{kij}\, \theta_i\, r_j$$

where ϵ_{ijk} is Levi-Civita's tensor which is totally anti-symmetric with $\epsilon_{123} = 1$.

So

$$\langle \vec{r} + \vec{\delta r} | = \langle \vec{r} | e^{i \vec{\delta r} \cdot \vec{p} / \hbar}$$

$$\parallel$$

$$\langle \vec{r} + \vec{\theta} \times \vec{r} | = \langle \vec{r} | e^{i (\overrightarrow{\theta \times r}) \cdot \vec{p} / \hbar}.$$

Because ϵ_{ijk} is unchanged by a cyclic permutation,

$$(\overrightarrow{\theta \times r}) \cdot \vec{p} = \sum_{i,j,k=1}^{3} \epsilon_{kij}\, \theta_i\, r_j\, p_k$$

$$= \sum_{i,j,k=1}^{3} \epsilon_{ijk}\, \theta_i\, r_j\, p_k = \vec{\theta} \cdot (\vec{r} \times \vec{p}).$$

So

$$\langle \vec{r} + \vec{\theta} \times \vec{r} | = \langle \vec{r} | e^{i \vec{\theta} \cdot \vec{L} / \hbar}$$

where the angular momentum vector $\vec{L}$ is

$$L_i = \sum_{j,k=1}^{3} \epsilon_{ijk} r_j p_k$$

or

$$\vec{L} = \vec{r} \times \vec{p}.$$

The adjoint of the top equation is

$$| r + \theta \times r \rangle = e^{-i \vec{\theta} \cdot \vec{L} / \hbar} | \vec{r} \rangle.$$

So the unitary operator that rotates states by $\theta = \| \vec{\theta} \| = \sqrt{\vec{\theta} \cdot \vec{\theta}}$ in the right-handed sense is

$$U(\vec{\theta}) = e^{i \vec{\theta} \cdot \vec{L} / \hbar}.$$

This angular-momentum operator $\vec{L}$ pertains to orbital angular momentum. If the particle has spin, then the full angular momentum operator is

$\vec{J} = \vec{L} + \vec{S}$ where $\vec{S} = \frac{\hbar}{2} \vec{\sigma}$ for spin $\frac{\hbar}{2}$.