Schrödinger's Equation

We have seen that H and $\vec{p}$ generate displacements in time and space

$$i\hbar \frac{\partial}{\partial t} \langle \vec{x}, t | \psi \rangle = \langle \vec{x}, t | H | \psi \rangle$$

and

$$\frac{\hbar}{i} \vec{\nabla} \langle \vec{x}, t | \psi \rangle = \langle \vec{x}, t | \vec{p} | \psi \rangle .$$

In non-relativistic quantum mechanics, we subtract mc^2 from the true hamiltonian for a single particle and take the limit as $v/c \to 0$. So for a single free particle

$$H = H_{true} - mc^2$$

$$= \sqrt{c^2 p^2 + m^2 c^4} - mc^2$$

$$= mc^2 \sqrt{1 + \frac{p^2}{mc^2}} - mc^2$$

$$= \frac{p^2}{2m} + \text{terms we neglect.}$$

A single particle of mass m interacting with a potential $V(\vec{x})$ has

$$H = \frac{\vec{p}^{\,2}}{2m} + V(\vec{x}) .$$

Schrödinger's equation then is

$$i\hbar \frac{\partial}{\partial t} \langle \vec{x},t|\psi\rangle = \langle \vec{x},t|H|\psi\rangle$$

$$= \langle \vec{x},t| \frac{\vec{p}^{\,2}}{2m} + V(\vec{x})\, |\psi\rangle$$

$$= \left[\left(\frac{\hbar}{i}\vec{\nabla}\right)^2 \frac{1}{2m} + V(\vec{x})\right] \langle \vec{x},t|\psi\rangle .$$

Δ is an abbreviation for $\vec{\nabla}^2$. So

$$i\hbar \frac{\partial}{\partial t} \langle \vec{x},t|\psi\rangle = \left[-\frac{\hbar^2}{2m}\Delta + V(\vec{x})\right] \langle \vec{x},t|\psi\rangle .$$

If $|\psi\rangle$ is an e-vec of H, with e-val E, then

$$i\hbar \frac{\partial}{\partial t} \langle \vec{x},t|\psi\rangle = \langle \vec{x},t|H|\psi\rangle = E \langle \vec{x},t|\psi\rangle$$

which we may integrate to

$$\langle x,t|\psi\rangle = e^{-iEt/\hbar} \langle \vec{x}|\psi\rangle$$

often written as

$$\psi(\vec{x},t) = e^{-iEt/\hbar}\, \psi(\vec{x}) .$$

For two particles interacting thru a potential $V(\vec{x}_1 - \vec{x}_2)$ the hamiltonian is

$$H = \frac{\vec{p}_1^{\,2}}{2m_1} + \frac{\vec{p}_2^{\,2}}{2m_2} + V(\vec{x}_1 - \vec{x}_2) ,$$

The momentum operator is

$$\vec{P} = \vec{p}_1 + \vec{p}_2 .$$

Notice that

$$[H, \vec{P}] = [V(\vec{x}_1 - \vec{x}_2), P] = 0$$

because

$$[\vec{x}_1 - x_2, \vec{P}] = [\vec{x}_1 - \vec{x}_2, \vec{p}_1 + \vec{p}_2]$$

$$= i\hbar - i\hbar = 0.$$

When H is time indepedent, the time-evolution operator is

$$U(t) = e^{-iHt/\hbar} \quad \text{and} \quad |\psi, t\rangle = U(t)|\psi\rangle$$

When H does involve time but $[H(t'), H(t'')] = 0$

then

$$|\psi, t\rangle = \exp\left[-i\int_0^t dt' \frac{H(t')}{\hbar}\right]|\psi\rangle,$$

so $\quad U(t) = \exp\left[-i\int_0^t dt'\, H(t')/\hbar\right]$, so

$$i\hbar\frac{\partial}{\partial t}\langle x|\psi, t\rangle = \langle x|H(t)|\psi\rangle,$$

When $[H(t'), H(t'')] \neq 0$, then we use Dyson's formula

$$U(t) = T\exp\left[-i\int_0^t dt'\, H(t')\right]$$

in which the H's are time ordered,

$$U(t) = \underbrace{e^{\frac{-i\,dt\,H(t)}{\hbar}} \cdots\cdots e^{\frac{-i\,dt\,H(dt)}{\hbar}}\, e^{-i\,dt\,H(0)/\hbar}}_{}$$

$\frac{t}{dt}$ factors with time increasing to the left.

$$\psi(\vec{x},t) = \langle \vec{x},t|\psi\rangle = \langle \vec{0},0| e^{i(P\cdot x - Ht)/\hbar} |\psi\rangle$$

So

$$|\vec{x},t\rangle = e^{-i(P\cdot x - Ht)/\hbar} |\vec{0},0\rangle .$$

So $\quad U(\vec{x},t) = e^{-i(P\cdot x - Ht)/\hbar}$

moves a state from $\vec{0},0$ to $\vec{x},t$.

So

$$e^{-i(P\cdot X - Ht)/\hbar} |\psi\rangle$$

actually moves the state $|\psi\rangle$ by $\vec{x},t$.

But

$$\langle \vec{0},0| e^{-i(P\cdot x - Ht)/\hbar} |\psi\rangle = \langle -\vec{x}, -t|\psi\rangle,$$

while

$$\langle \vec{0},0| e^{i(P\cdot x - Ht)/\hbar} |\psi\rangle = \langle \vec{x},t|\psi\rangle = \psi(\vec{x},t).$$

Most often in textbooks one sees

$$\langle \vec{x},t|\psi\rangle = \langle \vec{x}| e^{-iHt/\hbar} |\psi\rangle = \langle \vec{x}|\psi,t\rangle$$

But $e^{-iHt/\hbar}|\psi\rangle$ is really $|\psi\rangle$ moved backwards in time.

Look at picture on next page.

$$\langle \vec{x}+\vec{a}, t+b | \psi \rangle = \langle \vec{x}, t | \underbrace{e^{i(a \cdot P - bH)/\hbar} | \psi \rangle}_{\text{state moved by } (-\vec{a}, -b)}$$