

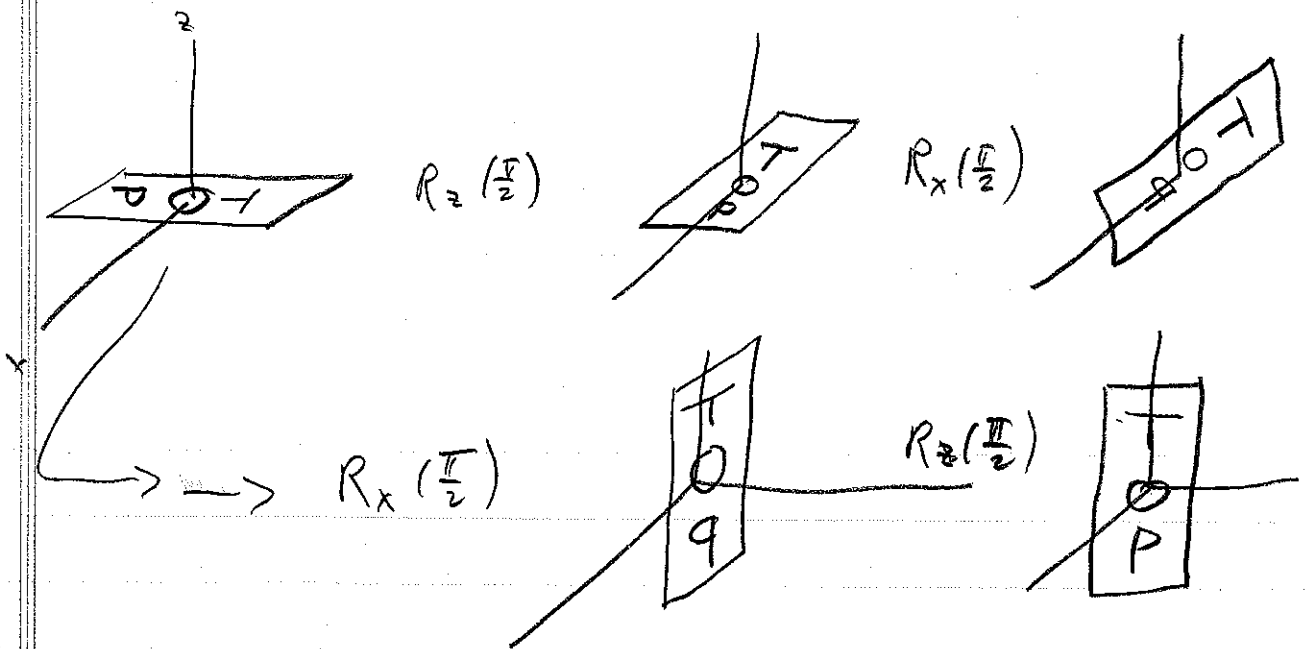
Chapter III

Angular Momentum and Rotations

$$\begin{pmatrix} V_x' \\ V_y' \\ V_z' \end{pmatrix} = R \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \quad R^T R = R R^T = 1 \quad \text{so}$$

$$\sum_{i=1}^3 V_i'^2 = \sum_{i,j,k=1}^3 (R_{ij} V_j) (R_{ik} V_k) = \sum_{j,k=1}^3 (R^T R)_{jk} V_j V_k$$

$$= \sum_{j,k=1}^3 \delta_{jk} V_j V_k = \sum_j V_j^2$$



$$R_z(\phi) = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_x(\epsilon) = \begin{pmatrix} 1 - \frac{\epsilon^2}{2} & -\epsilon & 0 \\ \epsilon & 1 - \frac{\epsilon^2}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_x(\epsilon) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - \frac{\epsilon^2}{2} & -\epsilon \\ 0 & \epsilon & 1 - \frac{\epsilon^2}{2} \end{pmatrix}$$

$$R_y(\epsilon) = \begin{pmatrix} 1 - \frac{\epsilon^2}{2} & 0 & \epsilon \\ 0 & 1 & 0 \\ -\epsilon & 0 & 1 - \frac{\epsilon^2}{2} \end{pmatrix}$$

$x \rightarrow y, y \rightarrow z, z \rightarrow x.$

$$R_x(\epsilon) R_y(\epsilon) = \begin{pmatrix} 1 - \frac{\epsilon^2}{2} & 0 & \epsilon \\ \epsilon^2 & 1 - \frac{\epsilon^2}{2} & -\epsilon \\ -\epsilon & \epsilon & 1 - \epsilon^2 \end{pmatrix}$$

$$R_y(\epsilon) R_x(\epsilon) = \begin{pmatrix} 1 - \frac{\epsilon^2}{2} & \epsilon^2 & \epsilon \\ 0 & 1 - \frac{\epsilon^2}{2} & -\epsilon \\ -\epsilon & \epsilon & 1 - \epsilon^2 \end{pmatrix}$$

So

$$\begin{aligned} R_x(\epsilon) R_y(\epsilon) - R_y(\epsilon) R_x(\epsilon) &= \begin{pmatrix} 0 & -\epsilon^2 & 0 \\ \epsilon^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= R_z(\epsilon^2) - I = R_z(\epsilon^2) - R_{\text{any}}(0). \end{aligned}$$

Do demo.

Like translations: $U(a) = e^{-i a \cdot p / \hbar}$
and

time evolution $U(t) = e^{-i t H / \hbar}$

so too

$$R_x(\theta) = e^{-i \theta J_x / \hbar}, \quad R_y(\theta) = e^{-i \theta J_y / \hbar}, \quad R_z(\theta) = e^{-i \theta J_z / \hbar}$$

So for small angles

$$R_y^{-1}(\epsilon) R_x^{-1}(\epsilon) R_y(\epsilon) R_x(\epsilon) = e^{i J_y \epsilon / \hbar} e^{i J_x \epsilon / \hbar} e^{-i J_y \epsilon / \hbar} e^{-i J_x \epsilon / \hbar}$$

$$\approx \left(1 + i \frac{J_y \epsilon}{\hbar} - \frac{\epsilon^2 J_y^2}{2 \hbar^2} \right) \left(1 + i \frac{J_x \epsilon}{\hbar} - \frac{\epsilon^2 J_x^2}{2 \hbar^2} \right)$$

$$\times \left(1 - i \frac{J_y \epsilon}{\hbar} - \frac{J_y^2 \epsilon^2}{2 \hbar^2} \right) \left(1 - i \frac{J_x \epsilon}{\hbar} - \frac{J_x^2 \epsilon^2}{2 \hbar^2} \right)$$

The terms involving only J_x (or only J_y) are the same as if J_y (or J_x) were absent; thus these terms cancel. So

$$R_y^{-1} R_x^{-1} R_y R_x \approx 1 - \frac{J_y J_x \epsilon^2}{\hbar^2} + \frac{J_y J_x \epsilon^2}{\hbar^2} + \frac{J_x J_y \epsilon^2}{\hbar^2} - \frac{J_y J_x \epsilon^2}{\hbar^2}$$

$$R_y^{-1}(\epsilon) R_x^{-1}(\epsilon) R_y(\epsilon) R_x(\epsilon) \approx 1 + \frac{\epsilon^2}{\hbar^2} [J_x, J_y]$$

Do Demo. Demo shows

$$R_y^{-1}(\epsilon) R_x^{-1}(\epsilon) R_y(\epsilon) R_x(\epsilon) \approx 1 + i \frac{\epsilon^2}{\hbar} J_z \approx R_z(\epsilon^2)$$

$$S_0 \quad [J_x, J_y] = i\hbar J_z$$

$$[J_a, J_b] = i\hbar \epsilon_{abc} J_c$$

$$|x\rangle_R = \mathcal{D}(R)|a\rangle$$

$$U_C = 1 - iG\epsilon$$

UPS

time

rotation

$$G = \frac{\vec{p} \cdot \hat{m}}{\hbar}$$

$$G = \frac{H}{\hbar}$$

$$G = \frac{\vec{J} \cdot \hat{n}}{\hbar}$$

eg.

$$\mathcal{D}_x(\phi) = \lim_{N \rightarrow \infty} \left(1 - i \frac{J_x \phi}{N \hbar} \right)^N = e^{-i \frac{J_x \phi}{\hbar}}$$

$$\approx 1 - i \frac{J_x \phi}{\hbar} - \frac{J_x^2 \phi^2}{2\hbar^2} + \dots$$

Groups

$$R \cdot 1 = R = 1 \cdot R \Rightarrow D(R) \cdot 1 = 1 \cdot D(R) = D(R)$$

$$R_1 R_2 = R_3 \Rightarrow D(R_1) D(R_2) = D(R_3)$$

$$R R^{-1} = R^{-1} R = 1 \Rightarrow D(R) D^{-1}(R) = D^{-1}(R) D(R) = 1$$

$$\begin{aligned} R \cdot (R_2 R_3) &= (R \cdot R_2) R_3 \Rightarrow D(R) (D(R_2) D(R_3)) \\ &= (D(R) D(R_2)) D(R_3) \\ &= D(R) D(R_2) D(R_3) \end{aligned}$$

$$\begin{aligned}
 & \left(1 - i \frac{J_x \epsilon}{\hbar} - \frac{J_x^2 \epsilon^2}{2\hbar^2} \right) \left(1 - i \frac{J_y \epsilon}{\hbar} - \frac{J_y^2 \epsilon^2}{2\hbar^2} \right) \\
 & - \left(1 - i \frac{J_y \epsilon}{\hbar} - \frac{J_y^2 \epsilon^2}{2\hbar^2} \right) \left(1 - i \frac{J_x \epsilon}{\hbar} - \frac{J_x^2 \epsilon^2}{2\hbar^2} \right) \\
 & = 1 - i \frac{J_z \epsilon^2}{\hbar} - 1 = -i \frac{J_z \epsilon^2}{\hbar}
 \end{aligned}$$

$O(\epsilon)$ -terms all cancel. $O(\epsilon^2)$ terms give

$$[J_x, J_y] = i\hbar J_z$$

$$[J_i, J_j] = \sum_{k=1}^3 i\hbar \epsilon_{ijk} J_k$$

These are the (non-abelian) commutation relations of $SO(3)$.

$SU(2)$

$$S_x = \frac{\hbar}{2} \begin{pmatrix} 1 + \chi - 1 & 1 - \chi + 1 \end{pmatrix} = S_1$$

$$S_y = \frac{\hbar}{2} \begin{pmatrix} -i | 1 + \chi - 1 & i | 1 - \chi + 1 \end{pmatrix} = S_2$$

$$S_z = \frac{\hbar}{2} \begin{pmatrix} 1 + \chi + 1 & 1 - \chi - 1 \end{pmatrix} = S_3$$

$$[S_i, S_j] = i\hbar \epsilon_{ijk} S_k$$

$$|\alpha\rangle_R = D_z(\phi) |\alpha\rangle$$

$$\langle S_1 \rangle(\phi) = \langle \alpha | S_1 | \alpha \rangle_R = \langle \alpha | D_z^\dagger(\phi) S_1 D_z(\phi) | \alpha \rangle$$

$$\langle S_1 \rangle(\phi) \equiv S_1(\phi) = \langle \alpha | e^{\frac{i S_3 \phi}{\hbar}} S_1 e^{-\frac{i S_3 \phi}{\hbar}} | \alpha \rangle$$

$$\frac{d S_1(\phi)}{d\phi} = \langle \alpha | e^{\frac{i S_3 \phi}{\hbar}} \frac{i}{\hbar} [S_3, S_1] e^{-\frac{i S_3 \phi}{\hbar}} | \alpha \rangle$$

$$= \frac{i}{\hbar} (i\hbar) \epsilon_{312} \langle \alpha | e^{\frac{i S_3 \phi}{\hbar}} S_2 e^{-\frac{i S_3 \phi}{\hbar}} | \alpha \rangle$$

$$= -S_2(\phi)$$

$$\frac{d S_2(\phi)}{d\phi} = \langle \alpha | e^{\frac{i S_3 \phi}{\hbar}} \frac{i}{\hbar} [S_3, S_2] e^{-\frac{i S_3 \phi}{\hbar}} | \alpha \rangle$$

$$= \langle \alpha | e^{\frac{i S_3 \phi}{\hbar}} \left(-\frac{\hbar}{i} \right) \epsilon_{321} S_1 e^{-\frac{i S_3 \phi}{\hbar}} | \alpha \rangle$$

$$= S_1(\phi)$$

So

$$\frac{dS_1(\phi)}{d\phi} = -S_2(\phi)$$

$$\frac{dS_2(\phi)}{d\phi} = S_1(\phi)$$

$$\frac{d^2}{d\phi^2} \begin{pmatrix} S_1(\phi) \\ S_2(\phi) \end{pmatrix} = - \begin{pmatrix} S_1(\phi) \\ S_2(\phi) \end{pmatrix}$$

$$S_1(\phi) = S_1(0) \cos \phi - S_2(0) \sin \phi$$

$$S_2(\phi) = S_1(0) \sin \phi + S_2(0) \cos \phi$$

which are (3.2.8-9). Clearly

$$S_3(\phi) = S_3(0) = \langle \alpha | S_3 | \alpha \rangle.$$

$$\langle S_i \rangle' / \hbar = R_{ij} / \hbar \langle S_j \rangle' / \hbar.$$

In general $\langle J_i \rangle' = R_{ij} \langle J_j \rangle'.$

But say $\phi = 2\pi$. Then

$$e^{-i S_3 \frac{2\pi}{\hbar}} (|+\rangle + |-\rangle) = e^{-i \frac{1}{2} \frac{2\pi}{\hbar}} |+\rangle$$

$$+ e^{i \frac{1}{2} \frac{2\pi}{\hbar}} |-\rangle = e^{-i\pi} |+\rangle + e^{i\pi} |-\rangle = -|+\rangle!$$

spin precession

$$e < 0 \quad H = - \frac{e}{mc} \mathbf{S} \cdot \mathbf{B} = \omega S_z$$

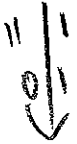
$$\omega = \frac{e \hbar B}{mc}$$

$$U(t, 0) = e^{-\frac{iHt}{\hbar}} = e^{-\frac{i\omega t S_z}{\hbar}}$$

$$\langle S_x \rangle_t = \langle S_x \rangle_0 \cos \omega t - \langle S_y \rangle_0 \sin \omega t$$

$$\langle S_y \rangle_t = \langle S_x \rangle_0 \sin \omega t + \langle S_y \rangle_0 \cos \omega t$$

$$\langle S_z \rangle_t = \langle S_z \rangle_0$$



$$m_\mu \approx 105 \text{ MeV}$$

$$\mu \rightarrow e + \nu_\mu + \bar{\nu}_e$$

\rightarrow P's are opposite to \vec{S}_μ

$$| \alpha, t=0; t \rangle = e^{-itH/\hbar} | \alpha, t=0; 0 \rangle$$

$$= e^{-\frac{i\omega t S_z}{\hbar}} (| +x + \alpha \rangle + | -x - \alpha \rangle)$$

$$= e^{\frac{-i\omega t}{2}} | +x + \alpha \rangle + e^{\frac{i\omega t}{2}} | -x - \alpha \rangle$$

So when $\omega t = 4\pi$, $| \alpha, t=0; t \rangle = | \alpha, t=0; 0 \rangle$

$$T_{\text{pre}} = \frac{2\pi}{\omega}$$

$$T_{\text{state}} = \frac{4\pi}{\omega}$$

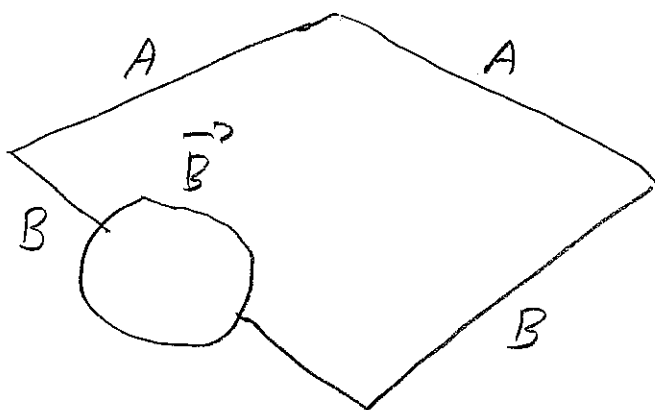
Neutron interferometry

due to charged quarks inside $n =$



$$H = \omega S_z = \frac{g_m e B}{m_p c} S_z$$

$$g_m \approx -1.91$$



$$\pm i\omega T/2$$

$$c_2 = c_2(B=0) e$$

where T is the time in \vec{B}

$$T = \frac{L}{v}$$

$$mv = p = \frac{h}{\lambda}$$

$$v = \frac{h}{m\lambda}$$

$$T = \frac{L m_p \lambda}{h} = \frac{4\pi}{\omega} = \frac{4\pi m_p c}{g_m e B}$$

successive maxima separated by $\Delta B = \frac{4\pi m_p c}{g_m e T} \approx$

$$\Delta B = \frac{4\pi c h}{e g_m \lambda L}$$

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$$\sigma_i \sigma_j = \delta_{ij} + i \sum_{k=1}^3 \epsilon_{ijk} \sigma_k$$

$$\text{tr } \sigma_i = 0$$

$$\begin{aligned} \sigma_i a \sigma_i b &= a_i b_i; \sigma_i \sigma_j = a_i b_j (\delta_{ij} + i \epsilon_{ijk} \sigma_k) \\ &= a \cdot b + i (a \times b) \cdot \vec{\sigma} \end{aligned}$$

$$e^{-i \frac{\vec{\sigma} \cdot \hat{n} \phi}{\hbar}} = e^{-i \frac{\vec{\sigma} \cdot \hat{n} \frac{\hbar}{\hbar} \phi}{\hbar}} = e^{-i \frac{\vec{\sigma} \cdot \hat{n} \phi}{2}}$$

$$(\sigma \cdot \hat{n})^2 = \hat{n} \cdot \hat{n} = 1 \qquad (\sigma \cdot \hat{n})^{2m} = 1$$

$$(\sigma \cdot \hat{n})^{2m+1} = \sigma \cdot \hat{n}$$

$$\begin{aligned} e^{-i \frac{\sigma \cdot \hat{n} \phi}{2}} &= \sum_{m=0}^{\infty} \frac{(-i\phi)^m}{m!} (\sigma \cdot \hat{n})^m \\ &= \sum_{k=0}^{\infty} \frac{(-i\phi)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(-i\phi)^{2k+1}}{(2k+1)!} \sigma \cdot \hat{n} \end{aligned}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(\frac{\phi}{2}\right)^{2k} - i \sigma \cdot \hat{n} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(\frac{\phi}{2}\right)^{2k+1}$$

$$= \cos \frac{\phi}{2} - i \sigma \cdot \hat{n} \sin \frac{\phi}{2}$$

$$e^{-i\frac{\vec{\sigma} \cdot \hat{n}}{2}\phi} = \begin{pmatrix} \cos \frac{\phi}{2} - i m_3 \sin \frac{\phi}{2} & (-i m_1 - m_2) \sin \frac{\phi}{2} \\ (-i m_1 + m_2) \sin \frac{\phi}{2} & \cos \frac{\phi}{2} + i m_3 \sin \frac{\phi}{2} \end{pmatrix}$$

$$\chi' = e^{-i\frac{\vec{\sigma} \cdot \hat{n}}{2}\phi} \chi$$

$$\begin{aligned} (\chi^\dagger \sigma_k \chi)' &= \chi^\dagger e^{i\frac{\vec{\sigma} \cdot \hat{n}}{2}\phi} \sigma_k e^{-i\frac{\vec{\sigma} \cdot \hat{n}}{2}\phi} \chi \\ &= R_{k\ell} \chi^\dagger \sigma_\ell \chi \end{aligned}$$

eg. $e^{i\frac{\sigma_3 \phi}{2}} \sigma_2 e^{-i\frac{\sigma_3 \phi}{2}} = \sigma_2 \cos \phi + \sigma_1 \sin \phi$

NB $e^{-i\frac{\vec{\sigma} \cdot \hat{n}}{2} 2\pi} = \cos \pi - i \vec{\sigma} \cdot \hat{n} \sin \pi = -1$

$$\vec{S} \cdot \hat{n} |\hat{n}, +\rangle = \frac{\hbar}{2} |\hat{n}, +\rangle$$

$$|\hat{n}, +\rangle = e^{-i\frac{\vec{\sigma} \cdot \hat{n}}{2}\theta} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\hat{m} = \frac{\hat{z} \times \hat{m}}{\sin \beta} \quad \theta = \beta \quad \phi = \alpha$$

$$\hat{n} = (\sin \beta \cos \alpha, \sin \beta \sin \alpha, \cos \beta)$$

$$\hat{z} \times \hat{m} = (-\sin \beta \sin \alpha, +\sin \beta \cos \alpha, 0)$$

$$\frac{\hat{z} \times \hat{m}}{\sin \beta} = (\sin \alpha, -\cos \alpha, 0)$$

$$-i \frac{\vec{\sigma}}{2} \cdot \beta (\sin \alpha, +\cos \alpha, 0)$$

$$|\hat{m}+\rangle \doteq e$$

$$= \left[\cos \frac{\beta}{2} - i(\sigma_1 \sin \alpha + \sigma_2 \cos \alpha) \sin \frac{\beta}{2} \right] \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\beta/2) \\ (+i \sin \alpha + \cos \alpha) \sin \frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} \cos \beta/2 \\ +e^{i\alpha} \sin \beta/2 \end{pmatrix}$$

$$= e^{i\alpha/2} \begin{pmatrix} \cos \beta/2 e^{-i\alpha/2} \\ \sin \beta/2 e^{i\alpha/2} \end{pmatrix}$$

$$|\hat{m}-\rangle = \left[\cos \frac{\beta}{2} - i(-\sigma_1 \sin \alpha + \sigma_2 \cos \alpha) \sin \frac{\beta}{2} \right] \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} (i \sin \alpha - \cos \alpha) \sin \frac{\beta}{2} \\ \cos \beta/2 \end{pmatrix} = \begin{pmatrix} -e^{-i\alpha} \sin \frac{\beta}{2} \\ \cos \beta/2 \end{pmatrix} = e^{-i\alpha/2} \begin{pmatrix} -\sin \beta/2 e^{i\alpha/2} \\ \cos \beta/2 e^{i\alpha/2} \end{pmatrix}$$

For spin 1 it is convenient to set

$$(J_k)_{ij} = i\hbar \epsilon_{ikj}$$

$$J_1 = i\hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = i\hbar \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

and

$$J_3 = i\hbar \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$J_1 J_2 = -\hbar^2 \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad J_2 J_1 = -\hbar^2 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[J_1, J_2] = -\hbar^2 \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = i\hbar J_3$$

In general

$$[J_i, J_j] = i\hbar \epsilon_{ijk} J_k.$$

So for spin 1 we may take

$$(J_k)_{ij} = i\hbar \epsilon_{ikj} \text{ which obey } [J_i, J_j] = i\hbar \epsilon_{ijk} J_k$$

in which none of the three generators is diagonal. This is the x, y, z representation.

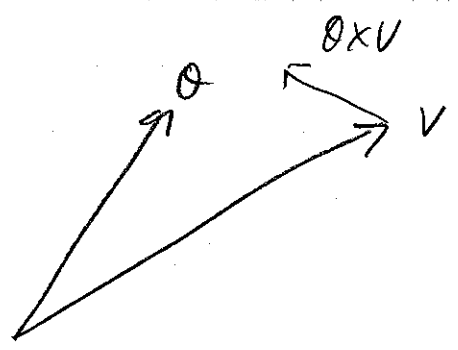
$$D(\vec{\theta}) = e^{-i\vec{\theta} \cdot \vec{J} / \hbar}$$

For small $|\vec{\theta}|$, $D(\vec{\theta}) \approx 1 - \frac{i\vec{\theta} \cdot \vec{J}}{\hbar}$ or

$$D(\vec{\theta})_{ij} \approx \left(1 - i\frac{\theta_k J_k}{\hbar}\right)_{ij} = \delta_{ij} - i\theta_k i\epsilon_{ikj} \\ \approx \delta_{ij} + \epsilon_{ikj} \theta_k$$

So under D , $V'_i = D(\vec{\theta})_{ij} V_j = V_i + \epsilon_{ikj} \theta_k V_j$

$$\vec{V}' = \vec{V} + \vec{\theta} \times \vec{V}$$



This is a right-handed rotation of V about $\vec{\theta}$ by $|\vec{\theta}|$.

Find $\vec{\theta}$ now.

$$A \equiv -i\frac{\theta \mathbf{J}}{\hbar} \quad A_{ij} = \epsilon_{ikj} \theta_k$$

$$A_{il}^2 = A_{ij} A_{jl} = \epsilon_{ikj} \theta_k \epsilon_{jml} \theta_m$$

$$= + \epsilon_{jik} \epsilon_{jml} \theta_k \theta_m$$

$$= (\delta_{im} \delta_{kl} - \delta_{il} \delta_{km}) \theta_k \theta_m$$

$$= \theta_i \theta_l - \delta_{il} \theta^2 \quad \text{So}$$

$$A_{im}^3 = A_{il}^2 A_{lm} = (\theta_i \theta_l - \delta_{il} \theta^2) \epsilon_{lkm} \theta_k$$

$$= -\theta^2 \epsilon_{ikm} \theta_k = -\theta^2 A_{im} \quad \text{So}$$

$$A^3 = -\theta^2 A \quad \text{So} \quad e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} \quad \text{or}$$

$$e^A = 1 + A + \frac{A^2}{2} - \frac{\theta^2 A}{3!} - \frac{\theta^2 A^3}{4!} + \frac{\theta^4 A}{5!} + \frac{\theta^4 A^3}{6!} - \frac{\theta^6 A}{7!}$$

The even terms are

$$1 + \frac{A^2}{2!} - \frac{\theta^2 A^2}{4!} + \frac{\theta^4 A^2}{6!} - \frac{\theta^6 A^2}{8!} + \dots$$

$$= 1 - \frac{A^2}{\theta^2} \left(-\frac{\theta^2}{2} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} \dots \right)$$

$$= 1 - \frac{A^2}{\theta^2} (-1 + \cos \theta)$$

The odd terms are

$$\begin{aligned}
 & A - \frac{\theta^2 A}{3!} + \frac{\theta^4 A}{5!} - \frac{\theta^6 A}{7!} + \frac{\theta^8 A}{9!} \dots \\
 & = \frac{A}{\theta} \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} \dots \right) = \frac{A}{\theta} \sin \theta.
 \end{aligned}$$

So

$$e^A = 1 - \frac{A^2}{2} (-1 + \cos \theta) + \frac{A}{\theta} \sin \theta.$$

Now $\left(\frac{A^2}{2}\right)_{ij} = \hat{\theta}_i \hat{\theta}_j - \delta_{ij}$ so

$$\begin{aligned}
 \left(e^{-i\theta \cdot \mathbf{J}/\hbar}\right)_{ij} &= (e^A)_{ij} = \delta_{ij} - (\hat{\theta}_i \hat{\theta}_j - \delta_{ij})(-1 + \cos \theta) \\
 &\quad + \frac{A_{ij}}{\theta} \sin \theta
 \end{aligned}$$

$$= \hat{\theta}_i \hat{\theta}_j - \cos \theta (\hat{\theta}_i \hat{\theta}_j - \delta_{ij}) + \epsilon_{ikj} \hat{\theta}_k \sin \theta$$

$$= \delta_{ij} \cos \theta + \epsilon_{ikj} \hat{\theta}_k \sin \theta + \hat{\theta}_i \hat{\theta}_j (1 - \cos \theta)$$

where $\theta = \sqrt{\hat{\theta} \cdot \hat{\theta}}$. Now $\hat{\theta}_i \hat{\theta}_j = I_{ij} + \left(\frac{-i\hat{\theta} \cdot \mathbf{J}}{\hbar}\right)_{ij}^2$, so

$$e^{-i\theta \cdot \mathbf{J}/\hbar} = I \cos \theta - i \frac{\hat{\theta} \cdot \mathbf{J}}{\hbar} \sin \theta + \left(I - \left(\frac{\hat{\theta} \cdot \mathbf{J}}{\hbar}\right)^2\right) (1 - \cos \theta)$$

$$= I - i \frac{\hat{\theta} \cdot \mathbf{J}}{\hbar} \sin \theta - (1 - \cos \theta) \left(\frac{\hat{\theta} \cdot \mathbf{J}}{\hbar}\right)^2 \text{ with } \epsilon_{ikj} \hat{\theta}_k = -i \left(\frac{\theta \cdot \mathbf{J}}{\hbar}\right)_{ij}$$

The general element of $SU(2)$ is

$$U(a, b) = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \quad \text{with} \quad |a|^2 + |b|^2 = 1.$$

Both $U(a, b)$ and $U(-a, -b)$ correspond to a single 3×3 matrix in $SO(3)$.

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Euler rotations

$$R(\alpha, \beta, \gamma) = R_z(\gamma) R_{y'}(\beta) R_z(\alpha) \quad \text{But}$$

$$R_{y'}(\beta) = R_z(\alpha) R_y(\beta) R_z^{-1}(\alpha) \quad \text{and}$$

$$R_z(\gamma) = R_{y'}(\beta) R_z(\gamma) R_{y'}^{-1}(\beta), \quad \text{and so}$$

$$\begin{aligned} R(\alpha, \beta, \gamma) &= R_{y'}(\beta) R_z(\gamma) R_{y'}^{-1}(\beta) R_z(\alpha) R_y(\beta) R_z^{-1}(\alpha) R_z(\alpha) \\ &= R_z(\alpha) R_y(\beta) R_z^{-1}(\alpha) R_z(\gamma) R_z(\alpha) R_y^{-1}(\beta) R_z^{-1}(\alpha) R_z(\alpha) R_y(\beta) \\ &= R_z(\alpha) R_y(\beta) R_z(\gamma) R_y^{-1}(\beta) R_y(\beta) \\ &= R_z(\alpha) R_y(\beta) R_z(\gamma) \quad \text{So} \end{aligned}$$

$$D(\alpha, \beta, \gamma) = D_z(\alpha) D_y(\beta) D_z(\gamma).$$

$su(2)$ case

$$D(\alpha, \beta, \gamma) = e^{-i\frac{\sigma_3}{2}\alpha} e^{-i\frac{\sigma_2}{2}\beta} e^{-i\frac{\sigma_3}{2}\gamma}$$

$$= \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} \begin{pmatrix} \cos \beta/2 & -i\sigma_2 \sin \beta/2 \\ \sin \beta/2 & \cos \beta/2 \end{pmatrix} \begin{pmatrix} e^{-i\gamma/2} & 0 \\ 0 & e^{i\gamma/2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} \begin{pmatrix} \cos \beta/2 & -\sin \beta/2 \\ \sin \beta/2 & \cos \beta/2 \end{pmatrix} \begin{pmatrix} e^{-i\gamma/2} & 0 \\ 0 & e^{i\gamma/2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} \begin{pmatrix} e^{-i\gamma/2} \cos \beta/2 & -e^{i\gamma/2} \sin \beta/2 \\ e^{-i\gamma/2} \sin \beta/2 & e^{i\gamma/2} \cos \beta/2 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i(\alpha+\gamma)/2} \cos \beta/2 & e^{i(\gamma-\alpha)/2} \sin \beta/2 \\ e^{-i(\alpha-\gamma)/2} \sin \beta/2 & e^{i(\alpha+\gamma)/2} \cos \beta/2 \end{pmatrix}$$

$$D_{m'm}^j(\alpha, \beta, \gamma) = \langle j, m' | e^{-i\frac{J_z}{\hbar}\alpha} e^{-i\frac{J_y}{\hbar}\beta} e^{i\frac{J_z}{\hbar}\gamma} | j, m \rangle$$

$$= e^{-i(m'\alpha + m\gamma)} \langle j, m' | e^{-i\frac{J_y}{\hbar}\beta} | j, m \rangle$$

ρ random spins from over.

Can't be $\frac{1}{2} (|+\rangle + |-\rangle)$.

Instead $\sum_j w_j = 1$

$$[A] = \sum_i w_i \langle \alpha^i | A | \alpha^i \rangle$$

$$= \sum_{ij} w_i \langle \alpha^i | A | a_j \rangle \langle a_j | \alpha^i \rangle$$

$$= \sum_{ij} w_i |\langle \alpha^i | a_j \rangle|^2 a_j$$

$$= \sum_{ijk} w_i \langle \alpha^i | b_j \rangle \langle b_j | A | b_k \rangle \langle b_k | \alpha^i \rangle$$

$$= \sum_{jk} \sum_i w_i \langle b_k | \alpha^i \rangle \langle \alpha^i | b_j \rangle \langle b_j | A | b_k \rangle$$

Let $\rho = \sum_i w_i | \alpha^i \rangle \langle \alpha^i |$

$$[A] = \sum_{jk} \langle b_k | \rho | b_j \rangle \langle b_j | A | b_k \rangle$$

$$= \sum_k \langle b_k | \rho A | b_k \rangle = \text{tr } \rho A.$$

$$1 = [1] = \text{tr } \rho = \sum_k \sum_i w_i \langle b_k | \alpha^i \rangle \langle \alpha^i | b_k \rangle$$

$$= \sum_i w_i \langle \alpha^i | \alpha^i \rangle = \sum_i w_i = 1.$$

$$\rho^\dagger = \sum_i w_i^* |\alpha^i \rangle \langle \alpha^i| = \rho$$

since the w_i 's are probabilities

But if $|4\rangle$ is any state, then the prob of finding $|4\rangle$ is

$$[|4\rangle\langle 4|] = \text{tr}[\rho |4\rangle\langle 4|] = \langle 4|\rho|4\rangle \geq 0$$

So $\rho > 0$. Thus $\rho^\dagger = \rho$, $\text{tr}\rho = 1$, $\rho > 0$ are the three properties of all density operators.

If $\rho = |\phi\rangle\langle\phi|$, then

$$\text{tr}\rho \quad \rho^2 = |\phi\rangle\langle\phi| |\phi\rangle\langle\phi| = |\phi\rangle\langle\phi| = \rho.$$

$$\rho(\rho - 1) = 0 \quad \text{tr}\rho^2 = \text{tr}\rho = 1.$$

$$\rho_{\text{diag}} = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

Nice examples in pp 179-181.

$$S_z^-: \quad \rho = |x\rangle\langle x| \\ = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$S_y^\pm: \quad \rho = \frac{1}{\sqrt{2}} (|+\rangle \pm i|-\rangle) \frac{1}{\sqrt{2}} (\langle +| \mp i\langle -|) \\ = \frac{1}{2} \begin{pmatrix} 1 \\ \pm i \end{pmatrix} \begin{pmatrix} 1, \mp i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & \mp i \\ \pm i & 1 \end{pmatrix}$$

$$\text{unpolarized:} \quad \rho = \frac{1}{2} \quad \text{tr} \rho \vec{S} = \vec{0}$$

$$\frac{1}{3} S_x, \quad \frac{1}{3} S_y, \quad \frac{1}{3} S_z :$$

$$\rho = \frac{1}{3} \frac{1}{\sqrt{2}} (|+\rangle + |-\rangle) \frac{1}{\sqrt{2}} (\langle +| + \langle -|) \\ + \frac{1}{3} \frac{1}{\sqrt{2}} (|+\rangle + i|-\rangle) \frac{1}{\sqrt{2}} (\langle +| - i\langle -|)$$

$$+ \frac{1}{3} |x\rangle\langle x|$$

$$= \frac{2}{3} |x\rangle\langle x| + \frac{1}{3} |-\rangle\langle -| + \frac{(1-i)}{6} |+\rangle\langle -| + \frac{(1+i)}{6} |-\rangle\langle +|$$

$$\text{tr} \rho = 1.$$

$$\text{tr} \rho S_x = \frac{\hbar}{2} \left(\frac{1-i}{6} + \frac{1+i}{6} \right) = \frac{\hbar}{6}$$

$$\text{tr} \rho S_y = \frac{\hbar}{2} \text{tr} \rho (-i|+\rangle\langle -| + i|-\rangle\langle +|) = \frac{\hbar}{2} \left[-i \frac{(1+i)}{6} + i \frac{(1-i)}{6} \right]$$

$$\text{tr } S_1 = \frac{t}{6}$$

$$\text{tr } S_2 = \frac{t}{2} \text{tr } \rho (1 + XH - 1 - X^{-1}) = \frac{t}{2} \left(\frac{2}{3} - \frac{1}{3} \right) = \frac{t}{6}$$

$$\begin{aligned} \rho(t) &= \sum_i w_i |d^{(i)}, t_0; t \rangle \langle \alpha^{(i)}, t_0; t| \\ &\quad - i \frac{H(t-t_0)}{t} \quad ; H(t-t_0)/t \\ &= e^{-i \frac{H(t-t_0)}{t}} \rho(0) e^{i \frac{H(t-t_0)}{t}} \end{aligned}$$

$$\frac{d\rho}{dt} = -\frac{i}{t} [H, \rho] \quad \leftrightarrow \quad \frac{\partial \rho}{\partial \tau} = -\{ \rho, H \}_{PB}$$

Liouville

Continuum ρ 's

$$[A] = \int d^3x' d^3x'' \langle x'' | \rho | x' \rangle \langle x' | A | x'' \rangle$$

$$\begin{aligned} \langle x'' | \rho | x' \rangle &= \langle x'' | \sum_i w_i | \alpha_i, x_i \rangle \langle x' | \\ &= \sum_i w_i \psi_i(x'') \psi_i^*(x') \end{aligned}$$

Stat. Mech.

class $\rho \doteq \frac{1}{N}$ order $\rho^2 = \rho; \rho = 1 + XH$

$$\sigma = -t(\rho \ln \rho)$$

$$= -\sum_k \rho_{kk} \ln(\rho_{kk}) \geq 0$$

Chaos $\sigma = -t \sum \frac{1}{N} \ln \frac{1}{N}$

$$= - \sum_{k=1}^N \frac{1}{N} \ln \frac{1}{N} = - \ln \frac{1}{N} = \ln N.$$

Order $\sigma = -t \sum p_{ii} \ln p_{ii}$

$$= - \sum_k p_{ii} \ln p_{ii}; p_{ii} = \begin{cases} 0 \\ 1 \end{cases}$$

$$\sigma = 0.$$

Entropy

$$S = k\sigma = -k t (\sum p \ln p).$$

Equilibrium

$$\frac{\partial F}{\partial t} = 0 \quad \text{so} \quad [F, H] = 0$$

$$U = t p H \quad \text{constant.}$$

Let's maximize S subject to the constraints

$$[F, H] = 0, \quad \sum p = 1, \quad \text{and} \quad t p H = U.$$

Use energy basis $H|i\rangle = E_i |i\rangle \quad \rho = \sum p_i |i\rangle \langle i|$

$$Q = -k \sum p_i \ln p_i + \lambda (\sum p_i - 1) + \mu (\sum p_i \epsilon_i - u)$$

$$Q = -k \sum_i p_i \ln p_i + \lambda (\sum p_i - 1) + \mu (\sum p_i \epsilon_i - u)$$

$$0 = -k \ln p_i - k + \lambda + \epsilon_i \mu$$

$$0 = \sum p_i - 1$$

$$0 = \sum p_i \epsilon_i - u$$

$$k \ln p_i = \lambda + \mu \epsilon_i - k$$

$$\ln p_i = \frac{1}{k} (\lambda + \mu \epsilon_i - k)$$

$$p_i = e^{(\lambda + \mu \epsilon_i - k)/k}$$

$$1 = \sum p_i = \sum_i e^{(\lambda + \mu \epsilon_i - k)/k}$$

$$\text{So } e^\lambda = \left[\sum_i e^{(\mu \epsilon_i - k)/k} \right]^{-1} \text{ and}$$

$$p_i = \frac{e^{(\mu \epsilon_i - k)/k}}{\sum_i e^{(\mu \epsilon_i - k)/k}} = \frac{e^{\mu \epsilon_i / k}}{\sum_i e^{\mu \epsilon_i / k}}$$

We let $\beta = \frac{1}{kT} = \frac{1}{kT}$ and then

$$p_i = \frac{e^{-\beta \epsilon_i}}{\sum_i e^{-\beta \epsilon_i}}$$

Let $Z = \sum p e^{-\beta M}$, Then

$$p = \frac{e^{-\beta M}}{Z(\beta)}.$$

$$[A] = \frac{\sum e^{-\beta M} A}{\sum e^{-\beta M}}$$

$$[E] = u = \frac{\sum e^{-\beta M} M}{\sum e^{-\beta M}} = -\frac{\partial}{\partial \beta} \ln \left(\sum e^{-\beta M} \right)$$

$$u = -\frac{\partial}{\partial \beta} \ln Z(\beta).$$

$$\beta = \frac{1}{kT}.$$

As $T \rightarrow \infty$ $p \rightarrow \frac{1}{N}$ chaos.

As $T \rightarrow 0$ $p \rightarrow 10 \times 01$ ground state.

$$\text{If } H = +\omega S_z$$

$$\rho = \frac{e^{-\beta \omega S_z}}{\int e^{-\beta \omega S_z}} = \frac{e^{-\beta \frac{\hbar \omega}{2} \sigma_z}}{\int e^{-\beta \frac{\hbar \omega}{2} \sigma_z}}$$

$$= \frac{\begin{pmatrix} e^{-\beta \hbar \omega / 2} & 0 \\ 0 & e^{\beta \hbar \omega / 2} \end{pmatrix}}{e^{\frac{\beta \hbar \omega}{2}} + e^{-\beta \hbar \omega / 2}}$$

So $\langle S_x \rangle = \langle S_y \rangle = 0$ but

$$\langle S_z \rangle = \frac{\hbar}{2} \left(\frac{e^{-\beta \hbar \omega / 2} - e^{\beta \hbar \omega / 2}}{e^{-\beta \hbar \omega / 2} + e^{\beta \hbar \omega / 2}} \right) = -\frac{\hbar}{2} \tanh \frac{\beta \hbar \omega}{2}$$

If

$$\frac{e}{m_0 c} [S_z] = \chi B, \text{ then}$$

$$\chi = \frac{|e| \hbar}{2 m_0 c B} \tanh \left(\frac{\beta \hbar \omega}{2} \right).$$

Evals and Evals of \vec{J}^2

$$J^2 = J_i^2$$

$$\begin{aligned} [J^2, J_j] &= J_i [J_i, J_j] + [J_i, J_j] J_i \\ &= \sum_{ik} (J_i i\hbar \epsilon_{ijk} J_k + i\hbar \epsilon_{ijk} J_k J_i) \end{aligned}$$

$$= i\hbar \sum_{ik} J_i J_k (\epsilon_{iik} + \epsilon_{kji})$$

$$= i\hbar \sum_{ik} J_i J_k (\epsilon_{ijk} - \epsilon_{ijk}) = 0.$$

So we diagonalize J^2 and J_3 .

$$J^2 |ab\rangle = a|ab\rangle$$

$$J_3 |ab\rangle = b|ab\rangle,$$

Let $J_{\pm} = J_1 \pm iJ_2$ ladders

$$[J_+, J_-] = [J_1 + iJ_2, J_1 - iJ_2]$$

$$= -i[J_1, J_2] + i[J_2, J_1] = -2i[J_1, J_2]$$

$$= -2i i\hbar \epsilon_{123} J_3 = 2\hbar J_3$$

$$[J_3, J_{\pm}] = [J_3, J_1 \pm iJ_2] = i\hbar \epsilon_{312} J_2 \pm i i\hbar \epsilon_{321} J_1$$

$$\begin{aligned}
 [J_3, J_{\pm}] &= i\hbar J_2 \mp \hbar \epsilon_{321} J_1 = i\hbar J_2 \pm \hbar J_1 \\
 &= \pm \hbar (J_1 \pm iJ_2) = \pm \hbar J_{\pm}
 \end{aligned}$$

$$[J^2, J_{\pm}] = 0.$$

$$\begin{aligned}
 J_3 (J_{\pm} |a, b\rangle) &= ([J_3, J_{\pm}] + J_{\pm} J_3) |a, b\rangle \\
 &= (\pm \hbar J_{\pm} + J_{\pm} J_3) |a, b\rangle \\
 &= (\pm \hbar J_{\pm} + J_{\pm} b) |a, b\rangle \\
 &= (b \pm \hbar) (J_{\pm} |a, b\rangle) \quad !
 \end{aligned}$$

Dispersion:

$$[J_3, J_{\pm}] = \pm \hbar J_{\pm} \text{ is like}$$

$$\begin{aligned}
 [x_i, \mathcal{T}(\vec{\ell})] &= [x_i, e^{\frac{-i\vec{p}\cdot\vec{\ell}}{\hbar}}] = \ell_i e^{\frac{-i\vec{p}\cdot\vec{\ell}}{\hbar}} \text{ (which is just} \\
 &e^{\frac{i\vec{p}\cdot\vec{\ell}}{\hbar}} \vec{x} e^{\frac{-i\vec{p}\cdot\vec{\ell}}{\hbar}} = \vec{x} + \vec{\ell}).
 \end{aligned}$$

$$\text{There are like } [a^\dagger a, a^\dagger] = a^\dagger, [a^\dagger a, a] = -a.$$

$$J^2 (J_{\pm} |a, b\rangle) = J_{\pm} J^2 |a, b\rangle = a (J_{\pm} |a, b\rangle)$$

So J_{\pm} changes b to $b \pm \hbar$ but leaves a alone.

$$J_{\pm} |a, b\rangle = c_{\pm} |a, b \pm \hbar\rangle.$$

We'll show that $a \geq b^2$.

$$\begin{aligned} \vec{J}^2 - J_3^2 &= J_1^2 + J_2^2 = \frac{1}{2} \{ J_1 + iJ_2, J_1 - iJ_2 \} \\ &= \frac{1}{2} \{ J_+, J_+^\dagger \} \end{aligned}$$

Now $\langle 4 | J_+^\dagger (J_+ + 14) \rangle \geq 0$ and $\langle 4 | J_+ (J_+^\dagger + 14) \rangle \geq 0$.

So

$$\begin{aligned} \langle a, b | \vec{J}^2 - J_3^2 | a, b \rangle &\geq 0 \\ \parallel \\ a - b^2 &\geq 0 \end{aligned}$$

So given a , there must be a b_{\max} and a b_{\min} such that

$$J_+ | a, b_{\max} \rangle = 0$$

and

$$J_- | a, b_{\min} \rangle = 0$$

Also

$$J_- J_+ | a, b_{\max} \rangle = 0$$

$$(J_1 - iJ_2)(J_1 + iJ_2) | a, b_{\max} \rangle = 0$$

$$= (J_1^2 + J_2^2 + i[J_1, J_2]) | a, b_{\max} \rangle$$

$$= (J_1^2 + J_2^2 - \hbar J_3) | a, b_{\max} \rangle$$

$$= (\vec{J}^2 - J_3^2 - \hbar J_3) | a, b_{\max} \rangle = (a - b^2 - \hbar b) | a, b_{\max} \rangle = 0$$

$$\text{So } a - b_{\max}^2 - t b_{\max} = 0 \quad \text{or}$$

$$a = b_{\max} (b_{\max} + t),$$

Also

$$0 = J_+ J_- |a, b'\rangle$$

$$b' = b_{\min}$$

$$= (J_1 + iJ_2)(J_1 - iJ_2) |a, b'\rangle$$

$$= (J_1^2 + J_2^2 - i[J_1, J_2]) |a, b'\rangle$$

$$= (J^2 - J_3^2 + t J_3) |a, b'\rangle$$

$$= (a - b'^2 + t b') |a, b'\rangle.$$

So

$$a = b_{\min} (b_{\min} - t).$$

$$b_{\max} (b_{\max} + t) = b_{\min} (b_{\min} - t)$$

$$b^2 + t b - b'(b' - t) = 0$$

$$b = \frac{-t \pm \sqrt{t^2 + 4b'(b' - t)}}{2} = \frac{-t \pm \sqrt{(2b' - t)^2}}{2}$$

$$b = \frac{-t \pm (2b' - t)}{2} = \begin{cases} b' - t \\ -b' \end{cases}$$

So either $b_{\max} = b_{\min} - t$ (absurd) or
 $b_{\max} = -b_{\min} \geq 0$

and $-b_{\max} \leq b \leq b_{\max}$.

Ladders $\Rightarrow b_{\max} = b_{\min} + n\hbar$

$$\text{So } b_{\max} = \frac{n\hbar}{2} \equiv j\hbar$$

So $j = \frac{n}{2}$ is $0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$

$$b_{\max} = j\hbar$$

$$\begin{aligned} a &= b_{\max}(b_{\max} + \hbar) = \hbar j(\hbar j + \hbar) \\ &= \hbar^2 j(j+1). \end{aligned}$$

Define $b = m\hbar$

m can be $-j, -j+1, \dots, j-1, j$ ($2j+1$)

$$\vec{J}^2 |j, m\rangle = j(j+1)\hbar^2 |j, m\rangle$$

$$J_z |j, m\rangle = m\hbar |j, m\rangle.$$

$$j = 0, \frac{1}{2}, 1, \dots$$

$m = -j, -j+1, \dots, j-1, j$: $2j+1$ states

matrix elements

$$\langle j' m' | \vec{J}^2 | j m \rangle = j(j+1) \hbar^2 \delta_{j'j} \delta_{m'm}$$

$$\langle j' m' | J_z | j m \rangle = m \hbar \delta_{j'j} \delta_{m'm}$$

$$\langle j m | J_+ J_+ | j m \rangle = \langle j m | J_- J_- | j m \rangle$$

$$= \langle j m | \vec{J}^2 - J_z^2 - \hbar J_z | j m \rangle = \hbar^2 [j(j+1) - m^2 - m]$$

$$= \hbar^2 [j(j+1) - m(m+1)] = \hbar^2 (j-m)(j+m+1)$$

Clearly

$$J_+ | j m \rangle = C_{j m}^+ | j, m+1 \rangle$$

$$|C_{j m}^+|^2 = \hbar^2 (j-m)(j+m+1)$$

So we may choose phases such that

$$J_+ | j m \rangle = \hbar \sqrt{(j-m)(j+m+1)} | j, m+1 \rangle.$$

$\frac{1}{2} \downarrow 99$

It follows that

$$\langle j m | J_- = \langle j, m+1 | (\hbar \sqrt{(j-m)(j+m+1)})$$

So

$$\langle j' m' | J_- | j m \rangle = \langle j' m'+1 | j m \rangle \hbar \sqrt{(j'-m')(j'+m'+1)}$$

$$= \delta_{j'j} \delta_{m'+1, m} \hbar \sqrt{(j'-m')(j'+m'+1)}$$

$$= \delta_{j'j} \delta_{m', m-1} \hbar \sqrt{(j-m+1)(j+m)}$$

$$J_- | j m \rangle = \hbar \sqrt{(j+m)(j-m+1)} | j, m-1 \rangle.$$

reps. of rots.

$$D_{m'm}^{(j)}(R) = \langle j'm' | D(R) | j'm \rangle = \langle j'm' | e^{-i \frac{J \cdot \hat{n} \phi}{\hbar}} | j'm \rangle$$

called Wigner functions.

$$J^2 D(R) = J^2 e^{-i \frac{J \cdot \hat{n} \phi}{\hbar}} = e^{-i \frac{J \cdot \hat{n} \phi}{\hbar}} J^2 = D(R) J^2$$

So

$$J^2 D(R) | j'm \rangle = D(R) J^2 | j'm \rangle = j(j+1) \hbar^2 D(R) | j'm \rangle$$

So $D(R)$ does not mix j 's.

$$\langle j'm' | D(R) | j'm \rangle$$

$$= \begin{pmatrix} [] \\ [] \\ [] \end{pmatrix}$$

is reducible; it

breaks up into 1×1 , 2×2 , 3×3 , etc. square matrices.

$D_{m'm}^{(j)}$ is the $(2j+1) \times (2j+1)$ block

called irreducible rep of rot group

$$(j) \quad D(1) = 1$$

$$[D^{(j)}(R)]^{-1} = D^{(j)}[R^{-1}] \quad (\phi \rightarrow -\phi)$$

$$\sum_{m'} D_{m'' m'}^{(j)}(R_1) D_{m' m}^{(j)}(R_2)$$

$$= \sum_{m'} \langle j m'' | D(R_1) | j m' \rangle \times \langle j m' | D(R_2) | j m \rangle$$

$$= \langle j m'' | D(R_1) D(R_2) | j m \rangle$$

$$= \langle j m'' | D(R_1 R_2) | j m \rangle$$

$$= D_{m'' m}^{(j)}(R_1 R_2)$$

$$D(R)^{-1} = \left[e^{-i J_z \frac{\phi}{\hbar}} \right]^{-1} = e^{i J_z \frac{\phi}{\hbar}}$$

$$= \left[e^{-i J_z \frac{\phi}{\hbar}} \right]^{\dagger} = D(R)^{\dagger}$$

So, $D(R)^{\dagger} = D(R)^{-1}$; $D(R)$ is unitary. Hence

$$D_{m'' m}^{(j)}(R^{-1}) = D_{m m'}^{(j)}(R)^*$$

||

$$\langle j m'' | D(R^{-1}) | j m \rangle = \langle j m' | D^{\dagger}(R) | j m \rangle = \langle j m | D(R) | j m' \rangle^*$$

Euler again

$$D_{m'm}^{(j)}(\alpha\beta\gamma) = \langle j m' | e^{-i\frac{J_3\alpha}{\hbar}} e^{-i\frac{J_2\beta}{\hbar}} e^{-i\frac{J_3\gamma}{\hbar}} | j m \rangle$$

$$= e^{-i(m'\alpha + m\gamma)} \langle j m' | e^{-i\frac{J_2\beta}{\hbar}} | j m \rangle$$

$$d_{m'm}^{(j)}(\beta) \equiv \langle j m' | e^{-i\frac{J_2\beta}{\hbar}} | j m \rangle.$$

$$\begin{array}{c} \uparrow 13 \\ 01 \\ \downarrow \end{array}$$
For $j = \frac{1}{2}$

$$e^{-i\frac{J_2\beta}{\hbar}} = \cos \frac{\beta}{2} - i \sigma_2 \sin \frac{\beta}{2}$$

$$d^{(\frac{1}{2})} = \begin{pmatrix} \cos \beta/2 & -\sin \beta/2 \\ \sin \beta/2 & \cos \beta/2 \end{pmatrix}$$

 $j = 1$

$$e^{-i\frac{\beta J_2}{\hbar}} = 1 - i\frac{J_2 \sin \beta}{\hbar} - (1 - \cos \beta) \frac{J_2^2}{\hbar^2}$$

which in the $|j m\rangle$ basis gives Eq. (3.5.57).

Orbital \vec{L}

$$\vec{L} = \vec{r} \times \vec{p}$$

$$[L_i, L_j] = [\epsilon_{irs} r_r p_s, \epsilon_{jke} r_k p_e]$$

$$\begin{aligned} [r_r p_s, r_k p_e] &= r_r [p_s, r_k p_e] + [r_r, r_k p_e] p_s \\ &= r_r (-i\hbar \delta_{sk}) p_e + i\hbar \delta_{rk} r_e p_s \end{aligned}$$

$$[L_i, L_j] = \epsilon_{irs} \epsilon_{jke} i\hbar (\delta_{re} r_k p_s - \delta_{sk} r_e p_r)$$

$$= i\hbar [\epsilon_{irs} \epsilon_{jkr} r_k p_s - \epsilon_{irs} \epsilon_{jse} r_e p_r]$$

$$= i\hbar [\epsilon_{irs} \epsilon_{jes} r_e p_r - \epsilon_{irs} \epsilon_{rjk} r_k p_s]$$

$$= i\hbar [(\delta_{ij} \delta_{re} - \delta_{ie} \delta_{rj}) r_e p_r - (\delta_{ij} \delta_{sk} - \delta_{ik} \delta_{sj}) r_k p_s]$$

$$= i\hbar [\delta_{ij} \vec{r} \cdot \vec{p} - r_j p_i - \delta_{ij} r \cdot p + r_i p_j]$$

$$= i\hbar (r_i p_j - r_j p_i)$$

$$i\hbar \epsilon_{ijk} L_k = i\hbar \epsilon_{ijk} \epsilon_{k\ell m} r_\ell p_m$$

$$= i\hbar (\delta_{ie} \delta_{jm} - \delta_{im} \delta_{je}) r_e p_m$$

$$= i\hbar (r_i p_j - r_j p_i)$$

Spherical coordinates

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times \frac{\hbar}{i} \vec{\nabla}$$

$$\vec{\nabla} = \hat{r} \frac{\partial}{\partial r} + \frac{\hat{\theta}}{r} \frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r \sin \theta} \frac{\partial}{\partial \phi}$$

$$\vec{L} = r \hat{r} \times \frac{\hbar}{i} \left[\hat{r} \frac{\partial}{\partial r} + \frac{\hat{\theta}}{r} \frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r \sin \theta} \frac{\partial}{\partial \phi} \right]$$

$$= \frac{\hbar}{i} \left(\hat{r} \times \hat{\theta} \frac{\partial}{\partial \theta} + \frac{\hat{r} \times \hat{\phi}}{\sin \theta} \frac{\partial}{\partial \phi} \right) = \frac{\hbar}{i} \left(\hat{\phi} \frac{\partial}{\partial \theta} - \frac{\hat{\theta}}{\sin \theta} \frac{\partial}{\partial \phi} \right)$$

$$\hat{r} = \frac{\vec{r}}{r} \quad \hat{\theta} = (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta)$$

$$\hat{\phi} = \hat{r} \times \hat{\theta} = (-\sin \phi, \cos \phi, 0) \quad \text{so}$$

$$\vec{L} = \frac{\hbar}{i} \left[(-\sin \phi, \cos \phi, 0) \frac{\partial}{\partial \theta} - (\cot \theta \cos \phi, \cot \theta \sin \phi, -1) \frac{\partial}{\partial \phi} \right]$$

$$L_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi} \quad L_x = -\frac{\hbar}{i} \left(\sin \phi \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \phi} \right)$$

$$L_y = \frac{\hbar}{i} \left(\cos \phi \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{\partial}{\partial \phi} \right)$$

$$L_+ = L_x + iL_y = \frac{\hbar}{i} \left[(i \cos \phi + i^2 \sin \phi) \frac{\partial}{\partial \theta} - (\cos \phi + i \sin \phi) \cot \theta \frac{\partial}{\partial \phi} \right]$$

$$L_+ = \frac{\hbar}{i} e^{i\phi} \left(i \frac{\partial}{\partial \theta} - \cot \theta \frac{\partial}{\partial \phi} \right)$$

$$L_- = -\frac{\hbar}{i} e^{-i\phi} \left(i \frac{\partial}{\partial \theta} + \cot \theta \frac{\partial}{\partial \phi} \right)$$

$$L_+ = \frac{\hbar}{i} e^{i\phi} \left(i \frac{\partial}{\partial \theta} - \cot \theta \frac{\partial}{\partial \phi} \right)$$

Now

$$\frac{\partial \hat{\phi}}{\partial \phi} = \frac{\partial}{\partial \phi} (-\sin \phi, \cos \phi, 0) = -(\cos \phi, \sin \phi, 0)$$

$$\frac{\partial \hat{\theta}}{\partial \theta} = \frac{\partial}{\partial \theta} (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta)$$

$$= -(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) = -\hat{n}$$

$$\frac{\partial \hat{\phi}}{\partial \theta} = 0$$

$$\hat{\theta} \cdot \frac{\partial \hat{\phi}}{\partial \phi} = -[\cos \theta \cos^2 \phi + \cos \theta \sin^2 \phi] = -\cos \theta$$

$$\frac{\partial \hat{\theta}}{\partial \phi} = \frac{\partial}{\partial \phi} (-\cos \theta \sin \phi, \cos \theta \cos \phi, 0) \text{ so } \hat{\theta} \cdot \frac{\partial \hat{\theta}}{\partial \phi} = 0$$

whence

$$\rightarrow L \cdot L = -\hbar^2 \left(\hat{\phi} \frac{\partial}{\partial \theta} - \frac{\hat{\theta}}{\sin \theta} \frac{\partial}{\partial \phi} \right) \cdot \left(\hat{\phi} \frac{\partial}{\partial \theta} - \frac{\hat{\theta}}{\sin \theta} \frac{\partial}{\partial \phi} \right)$$

$$= -\hbar^2 \left[\frac{\partial^2}{\partial \theta^2} - \frac{1}{\sin \theta} \left(\hat{\phi} \cdot \frac{\partial \hat{\theta}}{\partial \theta} \right) \frac{\partial}{\partial \phi} - \frac{1}{\sin \theta} \left(\hat{\theta} \cdot \frac{\partial \hat{\phi}}{\partial \phi} \right) \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right]$$

$$= -\hbar^2 \left(\frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right)$$

$$\text{So } [L_i, L_j] = i\hbar \epsilon_{ijk} L_k = i\hbar (v_i p_j - v_j p_i).$$

Now consider

$$1 - i \frac{\delta\phi}{\hbar} L_z = 1 - i \frac{\delta\phi}{\hbar} (x p_y - y p_x)$$

$$\left(1 - i \frac{\delta\phi}{\hbar} L_z\right) |x, y, z\rangle = |x - y \delta\phi, y + x \delta\phi, z\rangle$$

since

$$\left(1 - i \frac{\mathbf{a} \cdot \mathbf{p}}{\hbar}\right) |\vec{x}\rangle = |\vec{x} + \vec{a}\rangle$$

by (1.6.12) and (1.6.32). So $\left(1 - i \frac{\delta\phi L_z}{\hbar}\right)$ just rotates $|\vec{x}\rangle$ by $\delta\phi$ about \hat{z} .
counter-clockwise, right-handed rotation,

$$\langle x, y, z | \left(1 - i \frac{\delta\phi}{\hbar} L_z\right) |x\rangle = \langle x + y \delta\phi, y - x \delta\phi, z | x\rangle$$

is

$$\langle v, \theta, \phi | \left(1 - i \frac{\delta\phi}{\hbar} L_z\right) |x\rangle = \langle v, \theta, \phi - \delta\phi | x\rangle$$

$$\approx \langle v, \theta, \phi | x\rangle - \delta\phi \frac{\partial}{\partial\phi} \langle v, \theta, \phi | x\rangle$$

Thus

$$+i \frac{\hbar}{\hbar} L_z = \frac{\partial}{\partial\phi} \quad \text{or} \quad L_z = \frac{\hbar}{i} \frac{\partial}{\partial\phi}$$

Now \hat{x} :

$$\left(1 - i \frac{\delta\phi_x}{\hbar} L_x\right) |x, y, z\rangle = \left(1 - i \frac{\delta\phi_x}{\hbar} (y p_z - z p_y)\right) |x, y, z\rangle$$

$$= |x, y - \frac{\delta\phi_x}{\hbar} z, z + \frac{\delta\phi_x}{\hbar} y\rangle$$

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So

$$\langle x, y, z | (1 - i \frac{\delta\phi_x}{\hbar} L_x) | \alpha \rangle = \langle x, y + \delta\phi_x z, z - \delta\phi_x y | \alpha \rangle$$

Also

$$\begin{aligned} (1 - i \frac{\delta\phi_y}{\hbar} L_y) | x, y, z \rangle &= (1 - i \frac{\delta\phi_y}{\hbar} (z p_x - x p_z)) | x, y, z \rangle \\ &= | x + z \delta\phi_y, y, z - x \delta\phi_y \rangle \quad \text{So} \end{aligned}$$

$$\langle x, y, z | (1 - i \frac{\delta\phi_y}{\hbar} L_y) | \alpha \rangle = \langle x - z \delta\phi_y, y, z + x \delta\phi_y | \alpha \rangle \quad \text{So}$$

$$\begin{aligned} -i \frac{\delta\phi_x}{\hbar} \langle x, y, z | L_x | \alpha \rangle &= \langle x, y + \delta\phi_x z, z - \delta\phi_x y | \alpha \rangle - \langle x, y, z | \alpha \rangle \\ &= (0, z \delta\phi_x, -y \delta\phi_x) \cdot \vec{\nabla} \langle x, y, z | \alpha \rangle \end{aligned}$$

So

$$L_x = \frac{\hbar}{i} (0, -z, y) \cdot \vec{\nabla}$$

$$= \frac{\hbar}{i} r (0, -\cos\theta, +\sin\theta \sin\phi) \cdot \left[\hat{r} \frac{\partial}{\partial r} + \frac{\hat{\theta}}{r} \frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r \sin\theta} \frac{\partial}{\partial \phi} \right]$$

So since $\hat{y} = \hat{r} \sin\theta \sin\phi + \hat{\theta} \cos\theta \sin\phi + \hat{\phi} \cos\phi$ and $\hat{z} = \hat{r} \cos\theta - \hat{\theta} \sin\theta$ we have

$$L_x = \frac{\hbar}{i} r (0, -\cos\theta, \sin\theta \sin\phi) \cdot \left[\hat{r} \frac{\partial}{\partial r} + \frac{\hat{\theta}}{r} \frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r \sin\theta} \frac{\partial}{\partial \phi} \right]$$

$$r (0, -\cos\theta, \sin\theta \sin\phi) = r [-\cos\theta \hat{y} + \sin\theta \sin\phi \hat{z}]$$

$$\begin{aligned} &= r \cos\theta [\hat{r} \sin\theta \sin\phi + \hat{\theta} \cos\theta \sin\phi + \hat{\phi} \cos\phi] \\ &+ r \sin\theta \sin\phi [\hat{r} \cos\theta - \hat{\theta} \sin\theta] \end{aligned}$$

So

$$\begin{aligned}
 r(0, -\cos\theta, \sin\theta\sin\phi) &= \hat{\theta} \left(r\cos^2\theta\sin\phi - r\sin^2\theta\sin\phi \right) \\
 &\quad - r\cos\theta\cos\phi \hat{\phi} \\
 &= -r\hat{\theta}\sin\phi - r\cos\theta\cos\phi \hat{\phi}
 \end{aligned}$$

So

$$\begin{aligned}
 L_x &= \frac{\hbar}{i} r \left[-\hat{\theta}\sin\phi - \hat{\phi}\cos\theta\cos\phi \right] \cdot \left[\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial\phi} \right] \\
 &= \frac{\hbar}{i} \left[-\sin\phi \frac{\partial}{\partial\theta} - \cos\theta\cos\phi \frac{\partial}{\partial\phi} \right]
 \end{aligned}$$

Similarly using $\hat{x} = \hat{r}\sin\theta\cos\phi + \hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi$

$$\begin{aligned}
 -\frac{z}{\hbar} \delta\phi_y \langle xyzi | L_y | \alpha \rangle &= -\langle xyzi | \alpha \rangle + \langle x-z\delta\phi_y, y, z+x\delta\phi_y | \alpha \rangle \\
 &= (z\delta\phi_y, 0, +x\delta\phi_y) \cdot \vec{\nabla} \langle xyzi | \alpha \rangle
 \end{aligned}$$

So

$$\begin{aligned}
 L_y &= \frac{\hbar}{i} (z, 0, -x) \cdot \vec{\nabla} \\
 &= \frac{\hbar}{i} \left[r\cos\theta \hat{x} - r\sin\theta\cos\phi \hat{z} \right] \cdot \vec{\nabla} \\
 &= \frac{\hbar}{i} \left[r\cos\theta \left(\hat{r}\sin\theta\cos\phi + \hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi \right) \right. \\
 &\quad \left. - r\sin\theta\cos\phi \left(\hat{r}\cos\theta - \hat{\theta}\sin\theta \right) \right] \cdot \left[\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial\phi} \right]
 \end{aligned}$$

$$S_0$$

$$L_y = \frac{\hbar}{i} [\hat{\theta} n \cos \phi - \hat{\phi} r \cos \theta \sin \phi] \cdot \left(\frac{\hat{\theta}}{r} \frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r \sin \theta} \frac{\partial}{\partial \phi} \right)$$

$$= \frac{\hbar}{i} \left(\cos \phi \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{\partial}{\partial \phi} \right)$$

S_0

$$L_{\pm} = L_x \pm i L_y = \frac{\hbar}{i} \left[\left(-\sin \phi \pm i \cos \phi \right) \frac{\partial}{\partial \theta} + \cot \theta \left(-\cos \phi \mp i \sin \phi \right) \frac{\partial}{\partial \phi} \right]$$

$$= \frac{\hbar}{i} \left[\pm i (\cos \phi \pm i \sin \phi) \frac{\partial}{\partial \theta} - \cot \theta (\cos \phi \pm i \sin \phi) \frac{\partial}{\partial \phi} \right]$$

$$L_{\pm} = \frac{\hbar}{i} e^{\pm i \phi} \left(\pm i \frac{\partial}{\partial \theta} - \cot \theta \frac{\partial}{\partial \phi} \right)$$

Now

$$\langle \vec{x} | L^2 | \alpha \rangle = \langle \vec{x} | \frac{1}{2} L_+ L_- + \frac{1}{2} L_- L_+ + L_z^2 | \alpha \rangle$$

$$= \frac{1}{2} (-\hbar^2) e^{i\phi} \left(i \frac{\partial}{\partial \theta} - \cot \theta \frac{\partial}{\partial \phi} \right) e^{-i\phi} \left(-i \frac{\partial}{\partial \theta} - \cot \theta \frac{\partial}{\partial \phi} \right) \langle x | \alpha \rangle$$

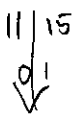
$$- \frac{\hbar^2}{2} e^{+i\phi} \left(-i \frac{\partial}{\partial \theta} - \cot \theta \frac{\partial}{\partial \phi} \right) e^{i\phi} \left(i \frac{\partial}{\partial \theta} - \cot \theta \frac{\partial}{\partial \phi} \right) \langle x | \alpha \rangle$$

$$- \hbar^2 \frac{\partial^2}{\partial \phi^2} \langle \vec{x} | \alpha \rangle.$$

$$\begin{aligned}
\langle x' | L^2 | \alpha \rangle &= -\hbar^2 \left\{ \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \cot^2 \theta \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial \varphi^2} \right\} \langle x' | \alpha \rangle \\
&= -\hbar^2 \left[\left(\frac{\cos^2 \theta}{\sin^2 \theta} + \frac{\sin^2 \theta}{\sin^2 \theta} \right) \frac{\partial^2}{\partial \varphi^2} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} \right] \langle x' | \alpha \rangle \\
&= -\hbar^2 \left[\frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} \right] \langle x' | \alpha \rangle.
\end{aligned}$$

Now

$$\begin{aligned}
L^2 &= (\mathbf{x} \wedge \mathbf{p})^2 = \epsilon_{ijk} \epsilon_{irs} x_j p_k x_r p_s \\
&= \begin{pmatrix} \delta_{jr} \delta_{ks} - \delta_{js} \delta_{kr} \end{pmatrix} x_j p_k x_r p_s \\
&= x_j p_k x_j p_k - x_j p_k x_k p_j \\
&= x_j (x_j p_k + [p_k, x_j]) p_k - x_j (x_k p_k + [p_k, x_k]) p_j \\
&= x^2 p^2 - i\hbar \mathbf{x} \cdot \mathbf{p} - x_j x \cdot \mathbf{p} p_j + 3i\hbar \mathbf{x} \cdot \mathbf{p} \\
&= x^2 p^2 + 2i\hbar \mathbf{x} \cdot \mathbf{p} - x_j (p_j x \cdot \mathbf{p} + [x \cdot \mathbf{p}, p_j]) \\
&= x^2 p^2 - (\mathbf{x} \cdot \mathbf{p})^2 + 2i\hbar \mathbf{x} \cdot \mathbf{p} - i\hbar \mathbf{x} \cdot \mathbf{p} \\
&= x^2 p^2 - (\mathbf{x} \cdot \mathbf{p})^2 + i\hbar \mathbf{x} \cdot \mathbf{p}
\end{aligned}$$

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$$\langle x' | \mathbf{x} \cdot \mathbf{p} | \alpha \rangle = x'_i \frac{\hbar}{i} \nabla'_i \langle x' | \alpha \rangle = \frac{\hbar}{i} \vec{r}' \cdot \left[\vec{r}' \frac{\partial}{\partial r'} + \dots \right] \langle x' | \alpha \rangle$$

$$\langle \vec{r} | \mathbf{x} \cdot \mathbf{p} | \alpha \rangle = \frac{\hbar}{i} r \frac{\partial}{\partial r} \langle \vec{r} | \alpha \rangle$$

And

$$\langle \vec{r} | (\mathbf{x} \cdot \mathbf{p})^2 | \alpha \rangle = \frac{\hbar}{i} r \frac{\partial}{\partial r} \frac{\hbar}{i} r \frac{\partial}{\partial r} \langle \vec{r} | \alpha \rangle$$

$$= -\hbar^2 \left(r^2 \frac{\partial^2}{\partial r^2} + r \frac{\partial}{\partial r} \right) \langle \vec{r} | \alpha \rangle$$

So

$$\begin{aligned} \langle \vec{r} | \vec{L}^2 | \alpha \rangle &= \langle \vec{r} | \mathbf{x}^2 \vec{p}^2 - (\mathbf{x} \cdot \mathbf{p})^2 + i\hbar \mathbf{x} \cdot \vec{p} | \alpha \rangle \\ &= r^2 \langle \vec{r} | \vec{p}^2 | \alpha \rangle + \hbar^2 \left(r^2 \frac{\partial^2}{\partial r^2} + 2r \frac{\partial}{\partial r} \right) \langle \vec{r} | \alpha \rangle \end{aligned}$$

whence

$$\langle \vec{r} | \frac{\vec{p}^2}{2m} | \alpha \rangle = -\frac{\hbar^2}{2m} \nabla^2 \langle \vec{r} | \alpha \rangle$$

$$= -\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) \langle \vec{r} | \alpha \rangle + \frac{1}{2m\hbar^2} \langle \vec{r} | \vec{L}^2 | \alpha \rangle$$

$$= -\frac{\hbar^2}{2m} \left[\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) \langle \vec{r} | \alpha \rangle - \frac{1}{\hbar^2 \hbar^2} \langle \vec{r} | \vec{L}^2 | \alpha \rangle \right]$$

As we saw in A.5, for spherically symmetric potentials energy eigenfunctions are of the form

$$\langle \vec{r} | n l m \rangle = R_{nl}(r) Y_{lm}(\theta, \phi)$$

(radial quantum number
or energy of free particle

This is because $-\frac{\hbar^2}{2m} \nabla^2 + V(r)$ is separable

in spherical coordinates. Equivalently

$$H = \frac{p^2}{2m} + V(r) \text{ so } [H, \vec{L}^2] = 0 \text{ so}$$

we can have simultaneous elets of H, \vec{L}^2, L_z .

Isolate $\langle \hat{n} | l m \rangle = \langle \theta, \phi | l m \rangle = Y_{lm}(\theta, \phi) \equiv Y_l^m(\theta, \phi)$

$|\hat{n}\rangle = |\theta, \phi\rangle$ is a direction eket.

$Y_l^m(\theta, \phi)$ is amplitude for $|l m\rangle$ to have θ, ϕ .

$$L_z |l, m\rangle = m \hbar |l, m\rangle$$

$$\langle \hat{n} | L_z |l, m\rangle = \frac{\hbar}{i} \frac{\partial}{\partial \phi} \langle \hat{n} | l m \rangle = \frac{\hbar}{i} \frac{\partial}{\partial \phi} Y_l^m(\theta, \phi)$$

$$\text{So } Y_l^m(\theta, \phi) = m \hbar \langle \hat{n} | l m \rangle = m \hbar Y_l^m(\theta, \phi)$$

$$\text{So } Y_l^m(\theta, \phi) \sim e^{im\phi} f(\theta).$$

$$\hat{L}^2 |l, m\rangle = l(l+1)\hbar^2 |l, m\rangle$$

$$\begin{aligned} \langle \hat{n} | \hat{L}^2 |l, m\rangle &= l(l+1)\hbar^2 Y_l^m(\theta, \phi) \\ &= -\hbar^2 \left[\frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} \right] Y_l^m(\theta, \phi) \end{aligned}$$

~

$$\left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} + l(l+1) \right] Y_l^m(\theta, \phi) = 0$$

or $[(\sin \theta)^{-1} \partial_\theta (\sin \theta \partial_\theta) - m^2 (\sin \theta)^{-2} + l(l+1)] Y_l^m(\theta, \phi) = 0$

$$\langle l', m' | l, m \rangle = \delta_{l'l} \delta_{m'm}$$

$$= \int d\Omega \langle l', m' | \hat{n} \times \hat{n} | l, m \rangle$$

$$= \int d\Omega Y_{l'}^{m'*}(\theta, \phi) Y_l^m(\theta, \phi)$$

$$= \int_{-1}^1 d\cos \theta \int_0^{2\pi} d\phi Y_{l'}^{m'*}(\theta, \phi) Y_l^m(\theta, \phi)$$

$$1 = \int d\Omega |\hat{n} \times \hat{n}| = \sum_{l=0}^{\infty} \sum_{m=-l}^l |l, m \rangle \langle l, m|$$

To get Y_l^m , try $m=l$

$$\langle \hat{n} | L_+ |l, l\rangle = 0$$

$$0 = \frac{\hbar}{i} e^{i\phi} \left(i \frac{\partial}{\partial \theta} - \cot \theta \frac{\partial}{\partial \phi} \right) Y_l^l(\theta, \phi)$$

since $Y_l^l(\theta, \phi) \sim e^{il\phi}$

$$\frac{\partial}{\partial \theta} Y_l^m(\theta, \phi) = \frac{1}{i} \cot \theta \frac{\partial}{\partial \phi} Y_l^m = l \cot \theta Y_l^m(\theta, \phi)$$

$$\frac{\partial}{\partial \theta} Y_l^m(\theta, \phi) = l \cot \theta Y_l^m(\theta, \phi)$$

$$\frac{dy}{d\theta} = l \cot \theta y \qquad \frac{dy}{y} = l \cot \theta d\theta$$

$$\ln y = l \int \cot \theta d\theta = l \int \frac{\cos \theta}{\sin \theta} d\theta = l \int \frac{d \sin \theta}{\sin \theta}$$

$$= l \ln \sin \theta = \ln \sin^l \theta$$

So

$$y \sim \sin^l \theta$$

$$Y_l^m(\theta, \phi) = c_l \sin^l \theta e^{im\phi} = \langle \theta \phi | l m \rangle$$

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In fact

$$c_l = \frac{(-1)^l}{2^l l!} \sqrt{\frac{(2l+1)(2l)!}{4\pi}}$$

Next

$$\langle \hat{n} | l, m-1 \rangle = \frac{\langle \hat{n} | l-1, m \rangle}{\hbar \sqrt{(l+m)(l-m+1)}}$$

$$\langle \vec{r} | l, m-1 \rangle = \frac{e^{-i\phi}}{\sqrt{(l+m)(l-m+1)}} \left(-\frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} \right) \langle \vec{r} | l, m \rangle$$

and so we get the result: For $m \geq 0$

$$Y_l^m(\theta, \phi) = \frac{(-1)^l}{2^l l!} \sqrt{\frac{(2l+1)(l+m)!}{4\pi(l-m)!}} \frac{e^{im\phi}}{\sin^m \theta} \frac{d^{l-m}}{d(\cos \theta)^{l-m}} (\sin \theta)^{2l}$$

and

$$Y_l^{-m}(\theta, \phi) = (-1)^m \left[Y_l^m(\theta, \phi) \right]^*$$

Always, any m

$$Y_l^m(\theta, \phi) \propto \sin^{|m|} \theta \text{ Poly}(\cos \theta)$$

(order $l-|m|$).

$m=0$:

$$Y_l^0(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos \theta)$$

For spin $\frac{1}{2}$

$$e^{i2\pi \frac{J_z}{\hbar}} |\alpha\rangle = e^{i2\pi m} |\alpha\rangle = -|\alpha\rangle$$

m half integer

but if

$$J = \hbar \times p \quad \text{then}$$

$$\langle \vec{r} | e^{-iJ_z \frac{2\pi}{\hbar}} |\alpha\rangle = \langle \vec{r} | \alpha \rangle,$$

So if $J = \hbar \times p$, then j must be an integer, not a half integer.

$Y_{l,m}$ & $D_{l,m}$

$$|\hat{n}\rangle = D(R)|\hat{z}\rangle = D(R)\sum_{l,m} |l,m\rangle \langle l,m|\hat{z}\rangle$$

$$\langle l,m'|\hat{n}\rangle = \sum_m \langle l,m'|D(R)|l,m\rangle \langle l,m|\hat{z}\rangle,$$

But

$$\langle l,m|\hat{z}\rangle = Y_l^m(\theta=0, \phi) \delta_{m,0} \quad \text{because}$$

$$\langle l,m|L_z|\hat{z}\rangle = 0 = m\hbar \langle l,m|\hat{z}\rangle, \quad \text{so}$$

$$\langle l,m|\hat{z}\rangle = \delta_{m,0} Y_l^0(\theta=0, \phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(1) \delta_{m,0}$$

Since $P_l(1) = 1$, this is

$$\langle l,m'|\hat{n}\rangle = \langle l,m'|D(R)|l,0\rangle \sqrt{\frac{2l+1}{4\pi}}$$

so

$$D_{m',0}^{(l)}(R) = \langle l,m'|\hat{n}\rangle \sqrt{\frac{4\pi}{2l+1}} = \sqrt{\frac{4\pi}{2l+1}} Y_l^{m'}(\theta, \phi)$$

any R that
takes \hat{z} to \hat{n} .

of \hat{n}

Euler case, for any γ ,

$$D_{m',0}^{(l)}(\phi, \theta, \gamma) = \sqrt{\frac{4\pi}{2l+1}} Y_l^{m'}(\theta, \phi). \quad \text{For } m=0$$

(See, A.5)

$$D_{0,0}^{(l)}(\phi, \theta, \gamma) = d_{0,0}^l(\theta) = P_l(\cos\theta) = \sqrt{\frac{4\pi}{2l+1}} Y_l^0(\theta, \phi) = P_l^0(\cos\theta).$$