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Time

$|\alpha, t_0; t\rangle \qquad t > t_0$

$$\lim_{t\to t_0} |\alpha\, t_0\, t\rangle = |\alpha\rangle = |\alpha\, t_0; t_0\rangle = |\alpha, t_0\rangle.$$

Time evolution $\quad |\alpha, t_0\rangle = |\alpha\rangle \longrightarrow |\alpha, t_0; t\rangle.$

Probability is conserved, so

$$|\alpha, t_0; t\rangle = \mathcal{U}(t, t_0)\, |\alpha, t_0\rangle$$

We expect that $u(t)|a_i\rangle$ still satisfy $\delta_{ij} = \langle a_i | a_j\rangle = \langle a_i | u^\dagger u | a_j\rangle$.

$$|\alpha, t_0\rangle = \sum_{a'} c_{a'}(t_0)\, |a'\rangle$$

$$|\alpha, t_0; t\rangle = \sum_{a'} c_{a'}(t)\, |a'\rangle$$

Expect

$$\sum_{a'} P(a', t) = \sum_{a'} |c_{a'}(t)|^2 = \sum_{a'} |c_{a'}(t_0)|^2 = \sum_{a'} P(a'; t_0)$$

unitarity

$$\langle \alpha\, t_0; t | \alpha, t_0; t\rangle = \langle \alpha\, t_0 | \alpha\, t_0\rangle = 1.$$

$$\mathcal{U}^\dagger(t, t_0)\, u(t, t_0) = 1$$

$$u(t_2, t_0) = u(t_2, t_1)\, u(t_1, t_0) \qquad t_2 > t_1 > t_0$$

$$|\alpha, t_0; t_0 + dt\rangle = u(t_0 + dt, t_0)\, |\alpha, t_0\rangle$$

$$\lim_{dt\to 0} u(t_0 + dt, t_0) = 1.$$

Let $\quad u(t_0 + dt, t_0) = 1 - i\Omega\, dt, \quad \Omega^\dagger = \Omega.$

Then if Ω is independent of time

$$U(t_0+dt_1+dt_2, t_0) = 1 - i\Omega(dt_1+dt_2)$$

$$= U(t_0+dt_1+dt_2, t_0+dt_1)\,U(t_0+dt_1, t_0)$$

$$= (1-i\Omega\,dt_2)(1-i\Omega\,dt_1).$$

$$U^\dagger(t_0+dt, t_0)\,U(t_0+dt, t_0) = (1+i\Omega^\dagger dt)(1-i\Omega\,dt)$$

$$= 1 - i(\Omega-\Omega^\dagger)\,dt = 1 \text{ if } \Omega = \Omega^\dagger.$$

The generator of time translation is the energy.

$$\Omega = \frac{H}{\hbar}$$

$$U(t_0+dt, t_0) = 1 - \frac{iH\,dt}{\hbar}, \text{ like } \mathcal{J}(dx) = 1 - \frac{i\vec{p}\cdot\vec{dx}}{\hbar}.$$

Same as $\vec{K} = \dfrac{\vec{P}}{\hbar}$, do get $\langle \dot{x} \rangle = \langle \dfrac{\vec{P}}{m} \rangle$.

Sch. Eq.

$$U(t+dt, t_0) = U(t+dt, t)\,U(t, t_0)$$

$$= \left(1 - \frac{iH\,dt}{\hbar}\right) U(t, t_0) \qquad \text{So}$$

$$U(t+dt, t_0) - U(t, t_0) = -\frac{iH\,dt}{\hbar}\,U(t, t_0).$$

or $$i\hbar \frac{\partial}{\partial t} \mathcal{U}(t,t_0) = H\,\mathcal{U}(t,t_0).$$

$$i\hbar \frac{\partial}{\partial t} \mathcal{U}(t,t_0)|\alpha,t_0\rangle = H\,\mathcal{U}(t,t_0)|\alpha,t_0\rangle$$

$$i\hbar \frac{\partial}{\partial t} |\alpha,t_0;t\rangle = H|\alpha,t_0;t\rangle$$

3 Cases.

1) $\frac{\partial H}{\partial t} = 0$ then $i\hbar \frac{\partial}{\partial t} \mathcal{U}(t,t_0) = H\,\mathcal{U}(t,t_0)$

is solved by
$$\mathcal{U}(t,t_0) = e^{-iH(t-t_0)/\hbar}$$

Proof:

$$i\hbar \underline{\left[\mathcal{U}(t+dt,t_0) - \mathcal{U}(t,t_0)\right]}_{dt} = i\hbar\left(e^{-iH dt/\hbar} - 1\right) \mathcal{U}(t,t_0)/dt$$

$$= i\hbar\left(1 - i\frac{H dt}{\hbar} - 1\right)\frac{\mathcal{U}(t,t_0)}{dt} = H\,\mathcal{U}(t,t_0).$$

2) $\frac{\partial H}{\partial t} \neq 0$ but $[H(t), H(t')] = 0$. Then by same argument
$$\mathcal{U}(t,t_0) = e^{-\frac{i}{\hbar}\int_{t_0}^{t} dt'\,H(t')}$$

(case 3) Now

$$U(t,t_0) = T\left\{ e^{-\frac{i}{\hbar}\int_{t_0}^{t} dt'\, H(t')} \right\}$$

$$= e^{-\frac{i}{\hbar} H(t)dt} \cdots \qquad e^{-\frac{i}{\hbar} H(t_0)dt}$$

So again

$$\frac{i\hbar}{dt}\left[U(t+dt,t_0) - U(t,t_0)\right] = \frac{i\hbar}{dt}\left(e^{-\frac{i}{\hbar}H(t)dt} - 1\right) U(t,t_0)$$

$$= \frac{i\hbar}{dt}\left(1 - \frac{i}{\hbar}H(t)dt - 1\right) U(t,t_0)$$

$$= H(t)\, U(t,t_0).$$

This $U(t,t_0)$ is given by the Dyson series

$$U(t,t_0) = 1 + \sum_{n=1}^{\infty} T\left\{ \left(-\frac{i}{\hbar}\int_{t_0}^{t} dt'\, H(t')\right)^n \right\} \frac{1}{n!}$$

$$= 1 + \sum_{n=1}^{\infty} \left(\frac{-i}{\hbar}\right)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n\, H(t_1) \cdots H(t_n).$$

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Suppose $[A, H] = 0$ then if A or H is nondeg. or if we choose it so that

$$H|a'\rangle = E_{a'}|a'\rangle.$$

Then

$$e^{-\frac{iHt}{\hbar}} = \sum_{a'} e^{-\frac{iHt}{\hbar}} |a'\rangle\langle a'| = \sum_{a'} e^{-\frac{iE_{a'}t}{\hbar}} |a'\rangle\langle a'| .$$

So if $|\alpha, t_0=0\rangle = \sum |a'\rangle\langle a'|\alpha\rangle = \sum c_{a'} |a'\rangle$, then

$$|\alpha, t_0=0; t\rangle = e^{-\frac{iHt}{\hbar}} |\alpha, t_0=0\rangle = \sum e^{-\frac{iE_{a'}t}{\hbar}} |a'\rangle\langle a'|\alpha\rangle .$$

So $c_{a'}(t) = e^{-\frac{iE_{a'}t}{\hbar}} c_{a'}(0) = e^{-\frac{iE_{a'}t}{\hbar}} c_{a'}$.

If $|\alpha, t_0=0\rangle = |a'\rangle$, then

A stationary state.

$$|\alpha, t_0=0; t\rangle = |a'\rangle e^{-iE_{a'}t/\hbar}$$

remains an e-ket of A and of H for all t,

Observables that commute with H are constant. If $|\alpha, t\rangle = e^{-\frac{iHt}{\hbar}}|\alpha\rangle$, then $\langle\alpha, t|A|\alpha, t\rangle = \langle\alpha| e^{\frac{iHt}{\hbar}} A e^{-\frac{iHt}{\hbar}}|\alpha\rangle = \langle\alpha|A|\alpha\rangle$. And even $\langle\beta, t|A|\alpha, t\rangle = \langle\beta|A|\alpha\rangle$.
If $[A, B] = [B, C] = 0$ etc. $[A, H] = [B, H] = 0$ etc.
then

$$e^{-\frac{iHt}{\hbar}} = \sum_{k'} |k'\rangle\langle k'| e^{-\frac{iE_{k'}t}{\hbar}}$$

$$K|k'\rangle = k'|k'\rangle$$

where $K = \{A\ B\ C\}$ $\quad k' = \{a' b' c' \ldots\}$.

$[A,H]=0 \qquad [B,H]\neq 0.$

$$|a', t_0=0; t\rangle = U(t,t_0)|a'\rangle$$

$$\langle B\rangle_t = \langle a'|U^\dagger(t,t_0)\, B\, U(t,t_0)|a'\rangle$$

$$= e^{\frac{iE_{a'}t}{\hbar}} \langle a'|B|a'\rangle e^{\frac{-iE_{a'}t}{\hbar}} = \langle a'|B|a'\rangle_{t_0}.$$

The state $|a'\rangle$ is stationary in that $\langle a'|B|a'\rangle$ is time independent even if $[B,H]\neq 0$.

Say $|\alpha, t_0=0\rangle = \sum_{a'} C_{a'}|a'\rangle$. Then

$$\langle B\rangle_t = \sum_{\substack{a'\\a''}} c^*_{a'} \langle a'| e^{\frac{iE_{a'}t}{\hbar}} B\, e^{\frac{-iE_{a''}t}{\hbar}} |a''\rangle C_{a''}$$

$$= \sum_{a'a''} e^{\frac{-i(E_{a''}-E_{a'})t}{\hbar}} C^*_{a'} C_{a''} \langle a'|B|a''\rangle$$

in which $\omega_{a''a'} = \frac{E_{a''}-E_{a'}}{\hbar}$ are Bohr frequencies.

Spin Precession

$$H = -\frac{e}{m_e c}\vec{S}\cdot\vec{B} \qquad (e<0)$$

B static, uniform, $\vec{B} = B\hat{z}$.

$$H = -\left(\frac{eB}{m_e c}\right)S_z .$$

$$H|\hat{z}\pm\rangle = \mp\left(\frac{eB}{m_e c}\right)\frac{\hbar}{2}|\hat{z}\pm\rangle$$

$$E_\pm = \mp\frac{e\hbar B}{2m_e c} .$$

Let $\omega = \frac{|e|B}{m_e c}$, so that $H = \omega S_z$.

$$U(t,0) = e^{-\frac{i\omega S_z t}{\hbar}}$$

$$|\alpha\rangle = c_+|+\rangle + c_-|-\rangle$$

$$|\alpha, t_0{=}0; t\rangle = U(t,0)|\alpha\rangle = e^{-i\omega\frac{S_z}{\hbar}t}\left(c_+|+\rangle + c_-|-\rangle\right)$$

$$= c_+ e^{-i\frac{\omega t}{2}}|+\rangle + c_- e^{i\frac{\omega t}{2}}|-\rangle$$

since $H|\pm\rangle = \pm\frac{\hbar\omega}{2}|\pm\rangle$.

If $|\alpha\rangle = |+\rangle$, $c_+ = 1$, $c_- = 0$, then

$$u(t,0)|+\rangle = e^{-i\frac{\omega t}{2}}|+\rangle \quad \text{for all } t$$

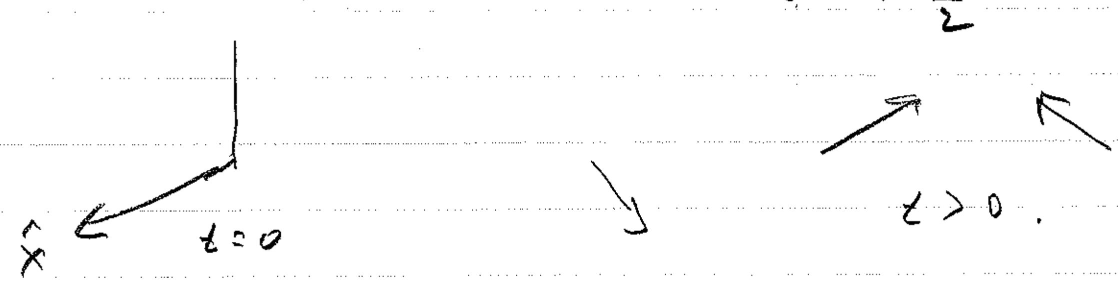
The state is stationary. All mean values of operators (without intrinsic, explicit time dependence) are constant in time. But if $|\alpha\rangle = |\hat{x}+\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$, then

$$|\alpha, t\rangle = e^{-i\omega t/2}\frac{|+\rangle}{\sqrt{2}} + \frac{e^{i\frac{\omega t}{2}}}{\sqrt{2}}|-\rangle, \qquad \mathcal{P}(\pm, t) = \frac{1}{2}$$

$$\mathcal{P}(\hat{x}\pm, t) = |\langle \hat{x}\pm|\alpha, t\rangle|^2$$

$$= \left|\left(\pm\frac{1}{\sqrt{2}}\langle -| \pm \frac{1}{\sqrt{2}}\langle +|\right)\left(e^{-i\frac{\omega t}{2}}\frac{|+\rangle}{\sqrt{2}} + \frac{e^{i\frac{\omega t}{2}}}{\sqrt{2}}|-\rangle\right)\right|^2$$

$$= \left|\frac{e^{-i\frac{\omega t}{2}}}{2} \pm \frac{e^{i\frac{\omega t}{2}}}{2}\right|^2 = \begin{cases} \cos^2\frac{\omega t}{2} & + \\ \sin^2\frac{\omega t}{2} & -. \end{cases}$$



$$\langle S_x\rangle = \langle \alpha, t|\frac{\hbar}{2}\left(|+\rangle\langle -| + |-\rangle\langle +|\right)|\alpha, t\rangle$$

$$= \frac{\hbar}{2}\left(\frac{e^{i\omega t}}{2} + \frac{e^{-i\omega t}}{2}\right) = \frac{\hbar}{2}\cos\omega t$$

ω here is $\omega_+ - \omega_- = \dfrac{E_+ - E_-}{\hbar}$ a Bohr frequency.

$$\langle S_y \rangle = \langle \alpha, t | \frac{\hbar}{2}\left(-i |+\rangle\langle -| + i |-\rangle\langle +|\right) | \alpha, t \rangle$$

$$= \frac{\hbar}{2}\left(-i \frac{e^{i\omega t}}{2} + i \frac{e^{-i\omega t}}{2}\right) = \frac{\hbar}{2} \sin \omega t$$

$$\langle S_z \rangle = \langle \alpha, t | \frac{\hbar}{2}\left(|+\rangle\langle +| - |-\rangle\langle -|\right) | \alpha, t \rangle$$

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$$= \frac{\hbar}{2}\left(\frac{1}{2} - \frac{1}{2}\right) = 0.$$

Correlation amplitude

$$C(t) = \langle \alpha | \mathcal{U}(t,0) | \alpha \rangle = \langle \alpha | \alpha, t_0 = 0; t \rangle.$$

If $H|\alpha\rangle = E_\alpha |\alpha\rangle$, then $C(t) = e^{\frac{-iE_\alpha t}{\hbar}}$, and

$|C(t)| = 1$ for all t in a stationary state.

If $H|a'\rangle = E_{a'}|a'\rangle$, and $|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle$,

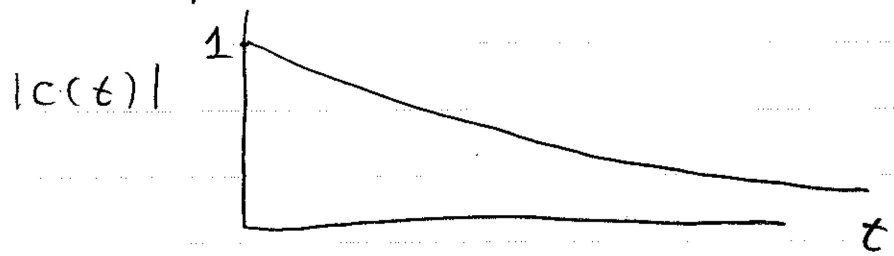
then

$$C(t) = \sum_{a'a''} \langle \alpha | a' \rangle \langle a' | e^{\frac{-iHt}{\hbar}} | a'' \rangle \langle a'' | \alpha \rangle$$

$$= \sum_{a'} \langle \alpha | a' \rangle \langle a' | \alpha \rangle e^{\frac{-iE_{a'}t}{\hbar}}$$

$$= \sum_{a'} e^{\frac{-iE_{a'}t}{\hbar}} |\langle a' | \alpha \rangle|^2 .$$

So we expect that $|C(t)|$ decreases from unity as t grows



Suppose $\sum_{a'} \to \int dE\, \rho(E)$ and $c_{a'} = \langle a'|\alpha\rangle \to g(E)$,

then

$$C(t) = \int dE\, |g(E)|^2 \rho(E)\, e^{-i\frac{Et}{\hbar}}$$ with

$$1 = C(0) = \int dE\, |g(E)|^2 \rho(E) = \langle\alpha|U(0,0)|\alpha\rangle.$$

So if $|g|^2\rho$ is peaked at E_0, then

$$C(t) = e^{-i\frac{E_0 t}{\hbar}} \int dE\, |g(E)|^2 \rho(E)\, e^{-i(E-E_0)t/\hbar}$$

and if

$$|g(E)|^2 \rho(E) = n_0\, e^{-\frac{(E-E_0)^2}{2\Delta E^2}}$$, then

$$C(t) = e^{-i\frac{E_0 t}{\hbar}}\, n_0 \int dE\, e^{-\frac{(E-E_0)^2}{2\Delta E^2} - i(E-E_0)t/\hbar}$$

$$= n_0\, e^{-i\frac{E_0 t}{\hbar}} \int dE\, e^{-\left(\frac{E-E_0}{\sqrt{2}\Delta E} + \frac{it\Delta E}{\sqrt{2}\hbar}\right)^2} e^{-\left(\frac{t\Delta E}{\sqrt{2}\hbar}\right)^2}$$

shift the contour now and get

$$C(t) = n_0 \, e^{-\frac{iE_0 t}{\hbar}} \, e^{-\left(\frac{t \Delta E}{\sqrt{2}\hbar}\right)^2} \int dE \, e^{-\frac{(E-E_0)^2}{2\Delta E^2}}$$

$$= n_0 \, e^{-\frac{iE_0 t}{\hbar}} \, e^{-\frac{\Delta E^2}{2\hbar^2}t^2} \int dE \, e^{-\frac{E^2}{2\Delta E^2}} \qquad x = \frac{E}{\sqrt{2}\,\Delta E}$$

$$\int dE \, e^{-\frac{E^2}{2\Delta E^2}} = \sqrt{2}\,\Delta E \int dx \, e^{-x^2} = \sqrt{2\pi}\,\Delta E$$

$$\left(\int dx \, e^{-x^2}\right)^2 = \int dx\,dy \, e^{-x^2-y^2} = \int_0^\infty 2\pi r\,dr \, e^{-r^2}$$

$$= \pi \int_0^\infty dx \, e^{-x} = \pi \qquad \text{So}$$

$$C(t) = \sqrt{2\pi}\,\Delta E \, n_0 \, e^{-\frac{iE_0 t}{\hbar}} \, e^{-\frac{\Delta E^2}{2\hbar^2}t^2}.$$

So when $\Delta t^2 \Delta E^2 = 2\hbar^2$, the correlation is down to $1/e$. So

$$\Delta t \approx \sqrt{2}\,\frac{\hbar}{\Delta E}.$$

In the spin case, with $|\alpha\rangle = |\hat{x}+\rangle$,

$|C(t)|^2 = |\langle \hat{x}+|U(t,0)|\alpha\rangle|^2 = \cos^2\frac{\omega t}{2}$ which is small when

$$\frac{\omega t}{2} = \frac{\pi}{2} \quad \text{or} \quad \pi = t\omega = t\frac{\Delta E}{\hbar} \quad \text{ie}$$

$$\Delta t \approx \pi \frac{\hbar}{\Delta E}.$$

But $\Delta E \Delta t \approx \hbar$ is not equivalent to $\Delta x \Delta p \geq \frac{\hbar}{2}$.

Unitary operators

1) change of basis $|b_i\rangle = U|a_i\rangle$, $U = \sum_i |b_i\rangle\langle a_i|$

2) change state $|\alpha\rangle \to U|\alpha\rangle$, $U = \mathcal{J}(d\vec{x}')$ or $U(t,t_0)$. In fact, these may be regarded as examples of (1).

N.B. $\langle\beta|\alpha\rangle \to \langle\beta|U^\dagger U|\alpha\rangle = \langle\beta|\alpha\rangle$.

$\langle\beta|Y|\alpha\rangle \to \left(\langle\beta|U^\dagger\right) Y \left(U|\alpha\rangle\right) = \langle\beta|(U^\dagger Y U)|\alpha\rangle$.

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associativity of matrix algebra

Two equivalent ways of thinking:

(1) $|\alpha\rangle \to U|\alpha\rangle$ $\quad Y \to Y$

or

(2) $Y \to U^\dagger Y U$ $\quad |\alpha\rangle \to |\alpha\rangle$.

The second approach is closer to classical physics where there are no states.

(1) $|\alpha\rangle \to \left(1 - i\frac{\vec{p}\cdot d\vec{x}'}{\hbar}\right)|\alpha\rangle$ $\qquad \vec{x} \to \vec{x}$

(2) $|\alpha\rangle \to |\alpha\rangle$ but $\vec{x} \to \left(1 + i\frac{p\cdot dx'}{\hbar}\right)\vec{x}\left(1 - i\frac{p\cdot dx'}{\hbar}\right)$

$$x_i \to x_i + \sum_{j=1}^{3} \frac{i\, dx_j'}{\hbar}\,[p_j, x_i] = x_i + \sum_j \frac{i}{\hbar}\, dx_j'(-i\hbar\delta_{ij})$$

$x_i \to x_i + dx_i'$ whence $\langle\vec{x}\rangle = \langle\vec{x} + d\vec{x}'\rangle$
$= \langle\vec{x}\rangle + \langle d\vec{x}'\rangle$.

Sch   Heis

Let $U(t,t_0) = U(t,0) = U(t) = e^{-iHt/\hbar}$. (Suppress t_0.)

$A^{(H)}(t) = U^\dagger(t)\, A^{(s)}\, U(t)$ $U(0) = 1$.

$A^{(H)}(0) = U^\dagger(0)\, A^{(s)} U(0) = A^{(s)}$

$|\alpha, t_0 = 0; t\rangle_H = |\alpha, t_0 = 0\rangle$ is independent of t

$|\alpha, t_0 = 0; t\rangle_S = U(t)\,|\alpha, t_0 = 0\rangle$.

${}_S\langle \alpha, t_0 = 0; t|\, A^{(s)}\, |\alpha, t_0 = 0, t\rangle_S = \langle \alpha, t_0 = 0|\, U^\dagger(t) A^{(s)} U(t)\, |\alpha, t_0 = 0\rangle$

$= {}_H\langle \alpha, t_0 = 0; t|\, A^{(H)}(t)\, |\alpha, t_0 = 0; t\rangle_H$

Sch   Heis.

$|\alpha, t\rangle_S = e^{-\frac{iHt}{\hbar}}|\alpha\rangle$ $|\alpha, t\rangle = |\alpha\rangle$
$= U(t)\,|\alpha\rangle$

$A^{(s)}(t) = A$ $A^H(t) = e^{\frac{iHt}{\hbar}} A\, e^{-\frac{iHt}{\hbar}}$

$= U^\dagger(t) A\, U(t)$

$A^H(t) = U^\dagger(t) A\, U(t)$ Assume $\frac{dA}{dt} = 0$.

$\frac{d}{dt} A^H(t) = \frac{d}{dt}U^\dagger(t)\, A U(t) + U^\dagger(t)\, A \frac{d}{dt} U(t)$.

$$U^\dagger(t) = e^{\frac{iHt}{\hbar}} \qquad \frac{dU^\dagger}{dt} = e^{\frac{iHt}{\hbar}} \frac{iH}{\hbar}$$

$$U = e^{-iHt/\hbar} \qquad \frac{dU}{dt} = \frac{-iH}{\hbar} e^{-iHt/\hbar}.$$

So

$$\frac{d}{dt} A^H(t) = U^\dagger(t) \frac{iH}{\hbar} A U(t) - U^\dagger(t) A \frac{iH}{\hbar} U(t)$$

$$= \frac{i}{\hbar} U^\dagger(t) [H, A] U(t)$$

$$= \frac{i}{\hbar} [H, U^\dagger(t) A U(t)] = \frac{i}{\hbar} [H, A^H(t)].$$

$$= \frac{1}{i\hbar} [A^H(t), H]. \qquad \text{H's eq. of mot.}$$

do problem 2.1

H's like the classical eq. (if $\partial A/\partial t = 0$)

$$\frac{dA}{dt} = \sum_i \frac{\partial A}{\partial q_i} \dot{q}_i + \frac{\partial A}{\partial p_i} \dot{p}_i$$

$$= \sum_i \frac{\partial A}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial H}{\partial q_i} = \{A, H\}_{P.B.}$$

using $[\,,\,] = i\hbar \{\,,\,\}_{PB}$ we get

$$\frac{dA}{dt} = \frac{1}{i\hbar} [A, H].$$

We may derive classical mechanics from QM by replacing $[\,,\,]$'s by $i\hbar \{\,,\,\}_{PB}$'s.

$$[x_i, F(\vec{p})] = i\hbar \{x_i, F(\vec{p})\} = i\hbar \frac{\partial F(p)}{\partial p_i}$$

$$[G(\vec{x}), p_i] = i\hbar \{G(\vec{x}), p_i\} = i\hbar \frac{\partial G(x)}{\partial x_i}$$

E.g. if $G = \vec{x}^2$

$$[G(\vec{x}), p_i] = \sum_{j=1}^{3} [x_j^2, p_i] = \sum_{j=1}^{3} x_j [x_j, p_i] + [x_j, p_i] x_j$$

$$= \sum_j 2i\hbar \delta_{ij} x_j = 2i\hbar x_i = i\hbar \frac{\partial G}{\partial x_i}.$$

Free particle:

Let $H = \frac{\vec{p}^2}{2m} = \sum_{i=1}^{3} \frac{p_i^2}{2m}$, Then

$$\frac{dp_i}{dt} = \frac{i}{\hbar}[H, p_i] = 0$$

$$\frac{dx_i}{dt} = \frac{i}{\hbar}[H, x_i] = \frac{i}{\hbar}\left[\frac{\vec{p}^2}{2m}, x_i\right] = \frac{i}{2\hbar m}(-i\hbar)\frac{\partial \vec{p}^2}{\partial p_i}$$

$$= \frac{1}{2m}\frac{\partial \vec{p}^2}{\partial p_i} = \frac{2p_i}{2m} = \frac{p_i}{m} = \frac{p_i(0)}{m}.$$

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So $x_i(t) = x_i(0) + \frac{p_i(0)}{m} t$.

N.B. $[x_i(0), x_j(0)] = 0$ But

$$[x_i(t), x_j(0)] = [x_i(0) + \frac{p_i(0)}{m}t, x_j(0)]$$

$$= \frac{t}{m}(-i\hbar)\delta_{ij} = -i\hbar\frac{t}{m}\delta_{ij}.$$

So by (1.4.53)

$$\langle (\Delta x_i(t))^2\rangle \langle (\Delta x_i(0))^2\rangle \geq \frac{1}{4}|\langle [x_i(t), x_i(0)]\rangle|^2$$

$$\geq \frac{\hbar^2}{4}\frac{t^2}{m^2}$$ or

$$\langle (\Delta x_i)^2\rangle_t \langle (\Delta x_i)^2\rangle_0 \geq \frac{\hbar^2 t^2}{4m^2}.$$ Interesting

$$H = \frac{\vec{p}^2}{2m} + V(\vec{x})$$

$$\frac{dp_i}{dt} = \frac{i}{\hbar}[H, p_i] = \frac{i}{\hbar}[V(x), p_i] = -\frac{\partial V(\vec{x})}{\partial x_i}$$

while still

$$\frac{dx_i}{dt} = \frac{i}{\hbar}[H, x_i] = \frac{p_i}{m}$$

So $\frac{d^2 x_i}{dt^2} = \frac{i}{\hbar}\left[H, \frac{dx_i}{dt}\right] = \frac{i}{\hbar}\left[H, \frac{p_i}{m}\right] = \frac{1}{m}\frac{dp_i}{dt}$

$$= \frac{i}{\hbar m}\, i\hbar \frac{\partial V}{\partial x_i} = -\frac{1}{m}\frac{\partial V}{\partial x_i}$$

So

$$m\frac{d^2 x_i}{dt^2} = \frac{dp_i}{dt} = -\frac{\partial V}{\partial x_i} \quad \text{or}$$

$$m\ddot{\vec{x}} = \dot{\vec{p}} = -\vec{\nabla} V(\vec{x}) \qquad \text{and so}$$

$$m\frac{d^2}{dt^2}\langle \vec{x} \rangle = \frac{d}{dt}\langle \vec{p} \rangle = -\langle \vec{\nabla} V(\vec{x}) \rangle$$

$$= -\vec{\nabla}\langle V(\vec{x}) \rangle.$$

No $\hbar$'s here. Centers of wave packets obey classical mechanics.

Sch: $A|a'\rangle = a'|a'\rangle$ so base kets, eigenkets of A don't move.

Heis: $A^H(t) = U^\dagger A(0) U$ so $A|a'\rangle = a'|a'\rangle$ implies

$U^\dagger A U U^\dagger |a'\rangle = a' U^\dagger |a'\rangle$, or $A^H(t)\, U^\dagger(t)|a'\rangle = a' U^\dagger(t)|a'\rangle$

So $U^\dagger(t)|a'\rangle$ is the base ket for the Heis rep.

$|a', t\rangle_H = U^\dagger(t)|a'\rangle$ is the H-picture basis ket.

$$i\hbar \frac{\partial}{\partial t}|a', t\rangle_H = -H\,|a', t\rangle_H$$

$$A^H(t)|a', t\rangle_H = a'\,|a', t\rangle_H$$

$|a', t\rangle_H$ is the state that has e-val a' at time t.
So the e-vals do not change with time.

Operators that are unitarily equivalent have the same e-vals.

$$A^H(t) = \sum a'\,|a', t\rangle\langle a', t|$$

$$= \sum a'\,U^\dagger(t)|a'\rangle\langle a'|U(t)$$

$$= U^\dagger A^{(s)} U = e^{\frac{iHt}{\hbar}} A\, e^{-i\frac{Ht}{\hbar}}.$$

Coefficients. $|\alpha, t=0; t\rangle = \sum |a'\rangle\langle a'|U(t)|\alpha, t=0\rangle$

$$c_{a'}(t) = \langle a'|U(t)\,|\alpha, t_0=0\rangle$$

$$= \Big(\langle a'|U(t)\Big)|\alpha, t_0=0\rangle$$

$$= {}_H\langle a', t|\alpha, t_0=0\rangle. = \Big(\langle a'|U(t)\Big)|\alpha, t_0=0\rangle.$$

$$\langle \vec{x}'|\alpha, t\rangle = \langle x'|\big(U(t)|\alpha\rangle\big) = \big(\langle x'|U(t)\big)|\alpha\rangle.$$

Sch   Heis

$A|a'\rangle = a'|a'\rangle \qquad B|b'\rangle = b'|b'\rangle$

at $t=0$ $\quad |\alpha\rangle = |a'\rangle$. then amplitude is

$$\langle b'|\big(U(t)|a'\rangle\big) = \big(\langle b'|U(t)\big)|a'\rangle$$

$$\text{Prob} = |\langle b'|U(t)|a'\rangle|^2.$$

One may say that $|a',t\rangle_H$

$$|a',t\rangle_H = U^\dagger(t)\,|a'\rangle = e^{\frac{iHt}{\hbar}}|a'\rangle$$

is the state that <u>at time t</u> is an e-ket of A with e-val a':

$$A\, e^{-\frac{iHt}{\hbar}}|a',t\rangle_H = A\,U(t)|a',t\rangle_H$$

$$= A\,U(t)\,U^\dagger(t)\,|a'\rangle = A|a'\rangle = a'|a'\rangle$$

$$= a'\,U(t)\,U^\dagger(t)|a'\rangle, = a'U(t)\,|a',t\rangle_H$$

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2$$

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left(x + \frac{ip}{m\omega} \right) = \frac{1}{\sqrt{2\hbar}} \left(\sqrt{m\omega}\, x + i \frac{p}{\sqrt{m\omega}} \right)$$

$$a^\dagger = \frac{1}{\sqrt{2\hbar}} \left(\sqrt{m\omega}\, x - i \frac{p}{\sqrt{m\omega}} \right) = \sqrt{\frac{m\omega}{2\hbar}} \left(x - \frac{ip}{m\omega} \right)$$

$$[a, a^\dagger] = \frac{1}{2\hbar} \Big(i[p, x] - i[x, p] \Big) = 1$$

$$N = a^\dagger a$$

$$a^\dagger a = \frac{m\omega}{2\hbar} \left(x - \frac{ip}{m\omega} \right) \left(x + \frac{ip}{m\omega} \right)$$

$$= \frac{m\omega}{2\hbar} \left(x^2 + \frac{p^2}{m^2\omega^2} + i \frac{[x, p]}{m\omega} \right)$$

$$\hbar\omega a^\dagger a = \frac{1}{2} m\omega^2 x^2 + \frac{p^2}{2m} - \frac{\hbar\omega}{2}$$ That is

$$H = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right) = \hbar\omega \left(N + \frac{1}{2} \right).$$

$$N|n\rangle = n|n\rangle \qquad H|n\rangle = \hbar\omega \left(n + \tfrac{1}{2} \right)|n\rangle.$$

$$E_n = \hbar\omega \left(n + \tfrac{1}{2} \right)$$

$$[N,a] = [a^\dagger a, a] = [a^\dagger, a]\, a = -a$$

$$[N,a^\dagger] = [a^\dagger a, a^\dagger] = a^\dagger [a, a^\dagger] = a^\dagger$$

$$N a^\dagger |n\rangle = \left([N, a^\dagger] + a^\dagger N \right) |n\rangle$$

$$= \left(a^\dagger + a^\dagger n \right) |n\rangle = (n+1)\, a^\dagger |n\rangle$$

$$N a |n\rangle = \left([N,a] + aN \right) |n\rangle = \left(-a + na \right) |n\rangle$$

$$= (n-1)\, a |n\rangle.$$

annihilation/creation lowering/raising

$$a|n\rangle = c|n-1\rangle$$

$$\langle n | a^\dagger a | n\rangle = |c|^2 = n$$ if states $\langle n|n\rangle = \langle n-1|n-1\rangle = 1$

Take $c > 0$ then

$$a|n\rangle = \sqrt{n}\, |n-1\rangle$$

So

$$\langle n | a^\dagger | m\rangle = \langle m | a | n\rangle^* = \sqrt{n}\, \langle m | n-1\rangle^* = \sqrt{n}\, \delta_{m,n-1}$$
$$= \sqrt{m+1}\, \delta_{m+1,n}$$

then $a^\dagger |m\rangle = \sqrt{m+1}\, |m+1\rangle$

$$a^\dagger |n\rangle = \sqrt{n+1}\, |n+1\rangle.$$

$a^2|n\rangle = \sqrt{n(n-1)}\,|n-2\rangle$ etc.

But $n = \langle n|a^\dagger a|n\rangle = (\langle n|a)^\dagger (a|n\rangle) \geq 0$.

So n must be a non-negative integer.

$E_0 = \frac{1}{2}\hbar\omega$ zero-point energy

$$a^\dagger|0\rangle = \sqrt{0+1}\,|1\rangle = |1\rangle$$

$$a^\dagger|1\rangle = \sqrt{1+1}\,|2\rangle = \sqrt{2}\,|2\rangle$$

$$|2\rangle = \frac{1}{\sqrt{2}}a^\dagger|1\rangle = \frac{1}{\sqrt{2}}(a^\dagger)^2|0\rangle.$$

$a^\dagger|2\rangle = \sqrt{2+1}\,|3\rangle = \sqrt{3}\,|3\rangle$ so

$$|3\rangle = \frac{1}{\sqrt{3}}a^\dagger|2\rangle = \frac{1}{\sqrt{3}}\frac{1}{\sqrt{2}}(a^\dagger)^2|1\rangle = = \frac{1}{\sqrt{3\cdot 2}}(a^\dagger)^3|1\rangle$$

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle.$$

$$E_n = (n+\tfrac{1}{2})\hbar\omega.$$

$$\langle n' | a | n \rangle = \sqrt{n}\, \delta_{n', n-1}$$

$$\langle n' | a^\dagger | n \rangle = \sqrt{n+1}\, \delta_{n', n+1}$$

$$x = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^\dagger) \qquad p = i\sqrt{\frac{m\hbar\omega}{2}}\,(a^\dagger - a)$$

So

$$\langle n' | x | n \rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\, \delta_{n', n-1} + \sqrt{n+1}\, \delta_{n', n+1}\right)$$

$$\langle n' | p | n \rangle = i\sqrt{\frac{m\hbar\omega}{2}}\left(-\sqrt{n}\, \delta_{n', n-1} + \sqrt{n+1}\, \delta_{n', n+1}\right).$$

Neither is diagonal in N-rep.

But neither x, p nor $a, a^\dagger$ commute with N.

$$a|0\rangle = 0$$

$$0 = \langle x' | a | 0 \rangle = \sqrt{\frac{m\omega}{2\hbar}}\,\langle x' | x + \frac{ip}{m\omega} | 0 \rangle$$

Thus

$$\frac{i}{m\omega}\langle x' | p | 0 \rangle = -x' \langle x' | 0 \rangle$$

$$\parallel$$

$$\frac{\hbar}{m\omega}\frac{d}{dx'}\langle x' | 0 \rangle = -x' \langle x' | 0 \rangle$$

$$\langle x' | 0 \rangle = \text{Const.}\, e^{-\frac{m\omega}{2\hbar}x'^2}$$

Let $x_0 = \sqrt{\frac{\hbar}{m\omega}}$. Then

$$\langle x' | 0 \rangle = \frac{1}{\sqrt{x_0\sqrt{\pi}}} e^{-\frac{1}{2}\left(\frac{x'}{x_0}\right)^2} .$$

$$\langle x' | 1 \rangle = \langle x' | a^\dagger | 0 \rangle \qquad a^\dagger = \frac{1}{\sqrt{2}\, x_0}\left(x - \frac{i\, p x_0^2}{\hbar} \right)$$

$$= \frac{1}{\sqrt{2}\, x_0} \langle x' | x - \frac{i}{\hbar} x_0^2 p | 0 \rangle$$

$$= \frac{1}{\sqrt{2}\, x_0} \left(x' - x_0^2 \frac{d}{dx'} \right) \langle x' | 0 \rangle$$

$$\langle x' | n \rangle = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} \frac{1}{x_0^{n+\frac{1}{2}}} \left(x' - x_0^2 \frac{d}{dx'} \right)^n e^{-\frac{1}{2}\left(\frac{x'}{x_0}\right)^2}$$

As an aside! $x^2 = \frac{\hbar}{2m\omega}(a + a^\dagger)^2$ and

$$\langle 0 | x^2 | 0 \rangle = \frac{x_0^2}{2} = \frac{\hbar}{2m\omega}$$

$$\langle 0 | p^2 | 0 \rangle = -\frac{m\hbar\omega}{2} \langle 0 | (a^\dagger - a)^2 | 0 \rangle = \frac{\hbar m\omega}{2}$$

So $\langle 0 | \frac{p^2}{2m} | 0 \rangle = \frac{\hbar\omega}{4} = \frac{\langle 0 | H | 0 \rangle}{2}$ and

$$\langle 0 | \frac{m\omega^2 x^2}{2} | 0 \rangle = \frac{\hbar\omega}{4} = \frac{\langle 0 | H | 0 \rangle}{2}$$

$V(\alpha r_1, \alpha r_2 \ldots \alpha r_n) = \alpha^k V(r_1 \ldots)$

$H = \frac{p_i^2}{2m} + V(\{r_i\})$

as in virial theorem: $\langle V \rangle = \frac{2E}{k+2}$ $\quad \langle T \rangle = \frac{kE}{k+2}$

90/02

$\langle 0|x|0\rangle = \langle 0|p|0\rangle = 0$

So $\quad (\Delta A \equiv A - \langle A\rangle)$

$$\langle 0|(\Delta x)^2|0\rangle = \langle 0|x^2|0\rangle = \frac{\hbar}{2m\omega}$$

$$\langle 0|(\Delta p)^2|0\rangle = \langle 0|p^2|0\rangle = \hbar\frac{m\omega}{2}$$

$$\langle 0|(\Delta x)^2|0\rangle\langle 0|(\Delta p)^2|0\rangle = \frac{\hbar^2}{4}.$$

$$\langle n|x^2|n\rangle = \langle n|\frac{\hbar}{2m\omega}(a+a^\dagger)^2|n\rangle = \frac{\hbar}{2m\omega}\langle n|aa^\dagger + a^\dagger a|n\rangle = \frac{(2n+1)\hbar}{2m\omega} = (n+\tfrac{1}{2})\frac{\hbar}{m\omega}$$

$$\langle n|p^2|n\rangle = -\frac{m\hbar\omega}{2}\langle n|(a^\dagger - a)^2|n\rangle = \frac{m\hbar\omega}{2}\langle n|a^\dagger a + aa^\dagger|n\rangle = (n+\tfrac{1}{2})\, m\hbar\omega$$ so

$$\langle n|(\Delta x)^2|n\rangle\langle n|(\Delta p)^2|n\rangle = (n+\tfrac{1}{2})^2\hbar^2.$$

Time evolution $\quad$ H's eq. $\Rightarrow$ (2.2.32, 33) $\Rightarrow$

$$\frac{dp}{dt} = -\frac{\partial V}{\partial x} = -m\omega^2 x \quad \text{and} \quad \frac{dx}{dt} = \frac{p}{m}$$

or more simply

$$\frac{da}{dt} = \frac{i}{\hbar}[H, a] = \frac{i}{\hbar}\left[\hbar\omega(a^\dagger a + \tfrac{1}{2}), a\right]$$

$$= i\omega a[a^\dagger, a] = -i\omega a$$

so $\quad a(t) = e^{-i\omega t} a(0)$

$$a^\dagger(t) = e^{i\omega t} a^\dagger(0).$$

Clearly $H(t) = \hbar\omega\left(a^\dagger(t)a(t) + \frac{1}{2}\right)$

$$= \hbar\omega\left(e^{i\omega t}a^\dagger a\, e^{-i\omega t} + \frac{1}{2}\right) = H(0).$$

Now $\quad x = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^\dagger)$ so

$$x(t) = \sqrt{\frac{\hbar}{2m\omega}}\left(a e^{-i\omega t} + a^\dagger e^{i\omega t}\right)$$

$$p(t) = i\sqrt{\frac{m\hbar\omega}{2}}\left(-a e^{-i\omega t} + a^\dagger e^{i\omega t}\right)$$

and since

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right) \qquad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right),$$

we get

$$x(t) = \frac{1}{2}\left[\left(x + \frac{ip}{m\omega}\right)e^{-i\omega t} + \left(x - \frac{ip}{m\omega}\right)e^{i\omega t}\right]$$

$$= x\cos\omega t + \frac{p}{m\omega}\sin\omega t$$

and

$$p(t) = \frac{im\omega}{2}\left[-\left(x + \frac{ip}{m\omega}\right)e^{-i\omega t} + \left(x - \frac{ip}{m\omega}\right)e^{i\omega t}\right]$$

$$p(t) = p\cos\omega t - m\omega x\sin\omega t$$

as one would expect classically.

One may also use the BH lemma

$$e^{G} A e^{-G} = A + [G,A] + \frac{1}{2!}[G[G A]] + \cdots \frac{1}{n!}[G[G \quad A]$$

$$e^{\frac{iHt}{\hbar}} x e^{-\frac{iHt}{\hbar}} = x + \frac{it}{\hbar}[H,x] + \left(\frac{it}{\hbar}\right)^2 \frac{1}{2!}[H,[H,x]]$$

$$[H,x] = -\frac{i\hbar p}{m} \qquad [H,p] = i\hbar m\omega^2 x$$

$$e^{\frac{iHt}{\hbar}} x e^{-\frac{iHt}{\hbar}} = x + \frac{p}{m}t - \frac{t^2\omega^2}{2!}x - \frac{1}{3!}t^3\omega^2\frac{p}{m} + \cdots$$

$$= (\cos\omega t)\, x + \frac{\sin\omega t}{m\omega}\, p.$$

Let $D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$. By BH

$$D(-\alpha)\, a\, D(\alpha) = D^{-1}(\alpha)\, a\, D(\alpha) = D^\dagger(\alpha)\, a\, D(\alpha)$$
$$= a + [\alpha^* a - \alpha a^\dagger, a] + \frac{1}{2!}[\alpha^* a - \alpha a^\dagger, [\alpha^* a - \alpha a^\dagger, a]] + \cdots$$

$$= a + [\alpha^* a - \alpha a^\dagger, a] = a - \alpha[a^\dagger, a] = a + \alpha$$

Now $a|0\rangle = 0$ So $D^{-1}(\alpha)\, a\, D(\alpha)|0\rangle = (a+\alpha)|0\rangle = \alpha|0\rangle$

or

$a D(\alpha)|0\rangle = \alpha D(\alpha)|0\rangle$. So $|\alpha\rangle = D(\alpha)|0\rangle$ is an e-ket of a with e-val $\alpha \in \mathbb{C}$.

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

Thus $|\langle n|\alpha\rangle|^2 = \frac{(|\alpha|^2)^n}{n!} e^{-|\alpha|^2}$ is a Poisson distribution.

$$D(\lambda\alpha) = e^{\lambda(\alpha a^\dagger - \alpha^* a)}, \qquad \lambda \text{ real}$$

Let

$$a(\lambda,\alpha) = D^\dagger(\lambda\alpha)\, a\, D(\lambda\alpha)$$

$$\frac{d\, a(\lambda,\alpha)}{d\lambda} = \frac{d}{d\lambda}\left[e^{-\lambda(\alpha a^\dagger - \alpha^* a)}\, a\, e^{\lambda(\alpha a^\dagger - \alpha^* a)} \right]$$

$$= D^\dagger(\alpha)\left[-(\alpha a^\dagger - \alpha^* a)a + a(\alpha a^\dagger - \alpha^* a) \right] D(\alpha)$$

$$= D^\dagger(\alpha)\, \alpha\, [a, a^\dagger]\, D(\alpha) = \alpha D^\dagger(\alpha) D(\alpha) = \alpha$$

So

$$a(\lambda,\alpha) = a + \lambda\alpha \quad \text{and}$$

$$a(1,\alpha) = D^\dagger(\alpha)\, a\, D(\alpha) = a + \alpha.$$

$$\langle \alpha | a(t) | \alpha \rangle = \langle \alpha | a e^{-i\omega t} | \alpha \rangle = e^{-i\omega t} \alpha$$

$$\langle \alpha | a^\dagger(t) | \alpha \rangle = \langle \alpha | a^\dagger e^{i\omega t} | \alpha \rangle = \alpha^* e^{i\omega t}$$

Note

$$\langle \beta | \alpha \rangle = e^{-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \beta^* \alpha}$$

$$1 = \int \frac{d^2\alpha}{\pi} |\alpha \rangle \langle \alpha| ,$$

A spin $\frac{1}{2}$ particle on a line in SUSY QM:

$$[H, Q_i] = 0 \qquad i = 1, 2$$

$$[Q_i, Q_j]_+ = \delta_{ij} H .$$

$$Q_1 \doteq \frac{1}{2} \left(\frac{\sigma_1}{\sqrt{m}} p + \sigma_2 \sqrt{m}\, W(x) \right)$$

$$Q_2 \doteq \frac{1}{2} \left(\frac{\sigma_2}{\sqrt{m}} p - \sigma_1 \sqrt{m}\, W(x) \right)$$

$$Q_i^\dagger = Q_i$$

The superpotential $|W(x)| \to \infty$ as $|x| \to \infty$

$$\langle x | \psi \rangle = \psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} \qquad \langle x | p | \psi \rangle = \frac{\hbar}{i} \frac{d\psi}{dx}$$

$$[Q_1, Q_1]_+ = 2Q_1^2 = H = \frac{p^2}{2m} + m\frac{W^2_{(x)}}{2} + \frac{\hbar}{2}\sigma_3 \frac{dW}{dx}$$

$$= [Q_2, Q_2]$$

$$[Q_1, Q_2]_+ = 0.$$

Susy is exact if $\quad Q_1|\psi\rangle = 0.$

Then

$$H|\psi\rangle = 2Q_1^2|\psi\rangle = 0. \quad \text{So since } H = 2Q_2^2$$

we have

$$0 = \langle\psi|Q_2^2|\psi\rangle = (\langle\psi|Q_2)^\dagger (Q_2|\psi\rangle) = 0 \quad \text{so } Q_2|\psi\rangle = 0 \text{ too.}$$

$$e^{-i(\theta_1 Q_1 + \theta_2 Q_2)}|\psi\rangle = |\psi\rangle.$$

$$H = 2Q_1^\dagger Q_1 \qquad \text{so all} \quad E_n's \geq 0.$$

$$Q_1|\psi\rangle = 0$$

$$\frac{1}{2}\left(\frac{\sigma_1}{\sqrt{m}} p + \sigma_2\sqrt{m}\, W(x)\right)|\psi\rangle = 0$$

$$\frac{\sigma_1}{m}\frac{\hbar}{i}\frac{d\psi}{dx} + \sigma_2 W\psi = 0$$

$$\frac{\hbar}{i}\frac{d\psi}{dx} = -m\sigma_1\sigma_2 W\psi = -im\sigma_3 W(x)\psi(x)$$

$$\hbar\frac{d\psi}{dx} = m\sigma_3 W\psi$$

$$\psi(x) = e^{\left(m\int_0^x \frac{dy}{\hbar} W(y)\sigma_3\right)}\psi(0).$$

10/4

$$\psi(\vec{x}',t) = \langle \vec{x}' | \alpha, t_0; t\rangle .$$

$$H = \frac{p^2}{2m} + V(\vec{x})$$

$$\langle \vec{x}'' | V(\vec{x}) | \vec{x}'\rangle = V(\vec{x}')\, \delta^3(\vec{x}' - \vec{x}'')$$

$$i\hbar \frac{\partial}{\partial t} \langle x' | \alpha, t_0; t\rangle = \langle x' | H | \alpha, t_0; t\rangle$$

$$= \langle x' | \frac{p^2}{2m} | \alpha, t_0; t\rangle + \langle x' | V(x) | \alpha, t_0; t\rangle$$

$$= -\frac{\hbar^2}{2m} \vec{\nabla}'^2 \langle x' | \alpha, t_0; t\rangle + V(\vec{x}') \langle x' | \alpha, t_0; t\rangle$$

or

$$i\hbar \frac{\partial}{\partial t} \psi(\vec{x}', t) = -\frac{\hbar^2}{2m} \vec{\nabla}'^2 \psi(\vec{x}', t) + V(\vec{x}')\, \psi(\vec{x}', t)$$

E. S. 1926.

Say $H|\alpha\rangle = E_\alpha |\alpha\rangle$. Then

$$|\alpha, 0; t\rangle = e^{-i\frac{Ht}{\hbar}} |\alpha, 0\rangle = e^{-i\frac{E_\alpha t}{\hbar}} |\alpha, 0\rangle$$

and so

$$i\hbar \frac{\partial}{\partial t} \langle x' | \alpha, t_0; t\rangle = i\hbar \left(-i\frac{E_\alpha}{\hbar}\right) \langle x' | \alpha, t_0; t\rangle$$

$$= E_\alpha \langle x' | \alpha, t_0; t\rangle .$$

So

$$E_\alpha \langle \vec{x}' | \alpha, t_0; t\rangle = -\frac{\hbar^2}{2m} \vec{\nabla}'^2 \langle \vec{x}' | \alpha, t_0; t\rangle + V(\vec{x}') \langle \vec{x}' | \alpha, t_0; t\rangle$$

Letting $\langle \vec{x}' | \alpha, t_0; t\rangle = e^{-i\frac{E_\alpha}{\hbar}(t - t_0)}\, \phi_{E_\alpha}(\vec{x}')$, we have

$$E\phi_E(\vec{x}') = -\frac{\hbar^2}{2m}\vec{\nabla}'^2\phi_E(\vec{x}') + V(\vec{x}')\phi_E(\vec{x}').$$

If $\quad E < \lim_{|\vec{x}'|\to\infty} V(\vec{x}')$,

then we expect

$$\phi_E(\vec{x}') \to 0 \quad \text{as} \quad |\vec{x}'|\to\infty.$$

The particle is confined and we expect that the time-independent S. eq. will have solutions that are normalizable only for discrete values of E.

If $V(\vec{x}) = V(r)$ is spherically symmetric, then as in A.5, we use spherical coordinates

$$E\,\psi_E(r,\theta,\phi) = -\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi_E(r\theta\phi)}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi_E(r,\theta,\phi)}{\partial\theta}\right)\right.$$

$$\left. + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi_E(r,\theta,\phi)}{\partial\phi^2}\right] + V(r)\,\psi_E(r,\theta,\phi)$$

We assume $\lim_{r\to 0} r^2 V(r) = 0$.

Let $\quad \psi_E(r,\theta,\phi) = R(r)\,Y_\ell^m(\theta,\phi)$ in which

$$-\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]Y_\ell^m(\theta,\phi) = \ell(\ell+1)\,Y_\ell^m(\theta,\phi)$$

for $\ell = 0, 1, 2, \cdots$ and $m = -\ell, -\ell+1, \cdots, \ell-1, \ell$

And

$$-i \frac{\partial}{\partial \phi} Y_\ell^m(\theta,\phi) = m\, Y_\ell^m(\theta,\phi).$$

and

$$\int d\Omega \; Y_{\ell'}^{m'*}(\theta,\phi)\, Y_\ell^m(\theta,\phi) = \delta_{\ell\ell'}\, \delta_{mm'}$$

Then

$$E\, R(r)\, Y_\ell^m(\theta,\phi) = -\frac{\hbar^2}{2m}\left[Y_\ell^m(\theta,\phi) \frac{1}{r^2}\frac{d}{dr} r^2 \frac{d}{dr} R(r) - \frac{\ell(\ell+1)}{r^2} R(r)\, Y_\ell^m(\theta,\phi) \right] + V(r)\, R(r)\, Y_\ell^m(\theta,\phi)$$

or

$$E\, R(r) = -\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{d}{dr} r^2 \frac{d}{dr} R(r) - \frac{\ell(\ell+1)}{r^2} R(r)\right] + V(r)\, R(r)$$

Let $u_E(r) = r\, R(r)$. Then

$$E\, u_E(r) = -\frac{\hbar^2}{2m}\frac{d^2 u_E(r)}{dr^2} + \left[V(r) + \frac{\ell(\ell+1)\hbar^2}{2mr^2}\right] u_E(r)$$

with $u_E(0) = 0$.

If $V(r) = 0$, $R(r) = c_\ell\, j_\ell(kr)$ (R not u_E)

where $k = \sqrt{\frac{2mE}{\hbar^2}}$ (free-particle case)

is a spherical Bessel function. These are not too bad

$$j_0(\rho) = \frac{\sin\rho}{\rho}, \qquad j_1(\rho) = \frac{\sin\rho}{\rho^2} - \frac{\cos\rho}{\rho}, \text{ etc}$$

$$j_\ell(\rho) \sim \frac{\rho^\ell}{(2\ell+1)!!} \quad (\rho \simeq 0)$$

If

$$V(r) = \begin{cases} 0 & r < R \\ \infty & r > R, \end{cases}$$

then

$$R_{n\ell}(r) = c_\ell\, j_\ell(\alpha_n r) \quad \text{with}$$

$$\alpha_n = \sqrt{\frac{2mE_n}{\hbar^2}} \quad \text{in which} \quad \alpha_n R = x_{n\ell}$$

is a root of the equation $0 = j_\ell(x_{n\ell})$.

H ~~atom~~ (Bryant)

$$V(r) = -\frac{Ze^2}{r} \qquad \rho = \left(\frac{8m_e|E|}{\hbar^2}\right)^{\frac{1}{2}} r$$

then

$$\psi_{n\ell m}(r\theta\phi) = R_{n\ell}(r)\, Y_\ell^m(\theta,\phi)$$

$$E_n = -\frac{1}{2} mc^2 \frac{Z^2\alpha^2}{n^2} = -\frac{1}{2} mc^2 \frac{Z^2}{n^2}\left(\frac{e^2}{\hbar c}\right)^2$$

where $\alpha = \frac{e^2}{\hbar c} = \frac{1}{137}$ in these units.

If
$$V(r) = \begin{cases} -V_0 & r < R \\ 0 & r > R, \end{cases}$$
then

for $r < R$
$$R(r) = A_\ell(r) = N j_\ell(\alpha r), \quad E < 0,$$
$$\alpha = \left[\frac{2m(V_0 - |E|)}{\hbar^2} \right]^{1/2}, \qquad N \text{ a constant.}$$

But for $r > R$
$$R(r) = A_\ell(r) = N_\ell \, h_\ell^{(1)}(ikr) \quad \text{where}$$
$$h_\ell^{(1)}(\rho) = j_\ell(\rho) + i n_\ell(\rho) \quad \text{and}$$
$$k = \left(\frac{2m|E|}{\hbar^2} \right)^{\frac{1}{2}}.$$

The Hankel function $h_\ell^{(1)}(\rho)$ as $\rho \to \infty$ is
$$h_\ell^{(1)}(\rho) \sim \frac{1}{\rho} e^{i[\rho - (\ell+1)\pi/2]}.$$

These are nice functions,
$$h_0^{(1)}(ikr) = -\frac{1}{kr} e^{-kr} \qquad h_1^{(1)}(ikr) = i\left(\frac{1}{kr} + \frac{1}{k^2 r^2} \right) e^{-kr}$$

$$Y_\ell^m(\theta,\varphi) = (-1)^m \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\; P_\ell^m(\cos\theta)\, e^{im\varphi} \qquad m \geq 0$$

$$P_\ell^m(x) = (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_\ell(x) \qquad m \geq 0$$

$$P_\ell(x) = \frac{(-1)^\ell}{2^\ell \ell!} \frac{d^\ell (1-x^2)^\ell}{dx^\ell}$$

for $m < 0$

$$Y_\ell^m(\theta,\varphi) = (-1)^{|m|} Y_\ell^{|m|*}(\theta,\varphi).$$

Always $Y_\ell^m(\ ,\varphi) \sim e^{im\varphi}$.

$$R_{n\ell}(r) = N_{n\ell}\, e^{-\rho/2} \rho^\ell L_{n+\ell}^{2\ell+1}(\rho) \qquad \rho = \frac{2Zr}{na_0}$$

$$N_{n\ell} = -\left\{ \left(\frac{2Z}{na_0}\right)^3 \frac{(n-\ell-1)!}{2n[(n+\ell)!]^3} \right\}^{\frac{1}{2}}$$ the minus sign is cosmetic

$a_0 = \frac{\hbar^2}{me^2}$ = Bohr radius. $\quad n \geq \ell+1$

$$E_n = \frac{-Z^2 e^2}{2n^2 a_0} \qquad L_p^q(\rho) = \frac{d^q}{d\rho^q} L_p(\rho)$$

$$L_p(\rho) = e^\rho \frac{d^p}{d\rho^p}\left(\rho^p e^{-\rho}\right) \qquad L_1(\rho) = e^\rho \frac{d}{d\rho}(\rho e^{-\rho}) = e^\rho(e^{-\rho} - \rho e^{-\rho})$$
$$= -\rho + 1$$

General properties of solutions of radial equation for bound states in spherically symmetric potentials:

$$E\,u(r) = -\frac{\hbar^2}{2m}u''(r) + \left[V(r) + \frac{\ell(\ell+1)\hbar^2}{2mr^2}\right]u(r)$$

If $V(r) \to 0$ as $r \to \infty$, then as $r \to \infty$

$$E\,u(r) \approx -\frac{\hbar^2}{2m}u''(r) \quad \text{or}$$

$$u(r) \sim e^{-\sqrt{\frac{2m|E|}{\hbar^2}}\,r}.$$

If $r^2 V(r) \to 0$ as $r \to 0$, then as $r \to 0$

$$E\,u(r) \approx -\frac{\hbar^2}{2m}u''(r) + \frac{\ell(\ell+1)\hbar^2}{2mr^2}u(r) \quad \text{or}$$

$$u''(r) = \frac{\ell(\ell+1)}{r^2}u(r) \qquad \text{let } u = r^\alpha$$

$$\alpha(\alpha-1) = \ell(\ell+1) \qquad \alpha^2 - \alpha - \ell(\ell+1) = 0$$

$$\alpha = \frac{1 \pm \sqrt{1+4\ell(\ell+1)}}{2} = \frac{1 \pm \sqrt{4\ell^2+4\ell+1}}{2} = \frac{1 \pm \sqrt{(2\ell+1)^2}}{2}$$

$$= \frac{1 \pm (2\ell+1)}{2} = \begin{cases} \ell+1 & \text{nonsingular choice} \\ -\ell \end{cases}$$

So as $r \to 0$ $\quad u(r) \approx r^{l+1}$

and as $r \to \infty$ $\quad u(r) \approx e^{-\sqrt{\frac{2m|E|}{\hbar^2}}\, r}$

So

$$R(r) = \frac{u(r)}{r} \sim r^{l} \qquad \text{as } r \to 0$$

and

$$R(r) = \frac{u(r)}{r} \sim \frac{e^{-\sqrt{\frac{2m|E|}{\hbar^2}}\, r}}{r} \qquad \text{as } r \to \infty.$$

Let $\rho = r/a_0$ $\qquad a_0 = \frac{\hbar^2}{m e^2}$ $\qquad m = \frac{m_e m_p}{m_e + m_p}$

$\lambda_{k,\ell} = \sqrt{-E_{k\ell}/E_I}$ $\qquad E_I = \frac{m e^4}{2\hbar^2}$

then

$$\left[\frac{d^2}{d\rho^2} - \frac{\ell(\ell+1)}{\rho^2} + \frac{2}{\rho} - \lambda_{k,\ell}^2\right] u_{k\ell}(\rho) = 0$$

Let

$$u_{k\ell}(\rho) = e^{-\rho\lambda_{k\ell}} \, y_{k\ell}(\rho) \qquad \text{then}$$

$$\left\{\frac{d^2}{d\rho^2} - 2\lambda_{k\ell}\frac{d}{d\rho} + \left[\frac{2}{\rho} - \frac{\ell(\ell+1)}{\rho^2}\right]\right\} y_{k\ell}(\rho) = 0$$

$y_{k\ell}(0) = 0$ $\qquad y_{k\ell}(\rho) = \rho^s \sum_{q=0}^{\infty} c_q r^q$

Find as before

$$s = \ell + 1.$$

and get

$$q(q+2\ell+1)c_q = 2[(q+\ell)\lambda_{k\ell} - 1]\, c_{q-1}$$

termination requires $\qquad \lambda_{q\ell} = \frac{1}{q+\ell}$ or $\lambda_{k\ell} = \frac{1}{k+\ell}$

$E_{k\ell} = -\frac{E_I}{(k+\ell)^2}$ $\qquad k + \ell = n$

Probability $\rho(x,t) = |\psi(x,t)|^2$

$$\frac{\partial\rho}{\partial t} = \psi^* \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi^*}{\partial t}$$

$$= \psi^* \left(-\frac{iH}{\hbar}\right)\psi + \psi \frac{\partial}{\partial t} \langle \psi | e^{\frac{iHt}{\hbar}} | x \rangle$$

$$= -\frac{i}{\hbar} \psi^* H \psi + \frac{i}{\hbar} \psi H \psi^*$$

$$= -\frac{i}{\hbar} \psi^*(x,t) \left(-\frac{\hbar^2}{2m} \nabla^2 + V(x) \right) \psi(x,t)$$

$$+ \frac{i}{\hbar} \psi(x,t) \left(-\frac{\hbar^2}{2m} \nabla^2 + V(x) \right) \psi^*(x,t)$$

$$= \frac{i\hbar}{2m} \left(\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^* \right)$$

$$= \frac{i\hbar}{2m} \vec{\nabla} \cdot \left(\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^* \right) \equiv -\nabla \cdot \vec{J}$$

$$\vec{J} = -\frac{i\hbar}{2m} \left(\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^* \right)$$

$$= \frac{\hbar}{m} \operatorname{Im} \left(\psi^* \nabla \psi \right)$$

$$0 = \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J}$$

$$\int d^3x\, j(x,t) = \int d^3x\, \frac{-i\hbar}{2m}\left(\psi^* \nabla\psi - \psi \nabla\psi^*\right)$$

$$= \int d^3x\, \frac{-i\hbar}{m} \psi^* \nabla \psi = \int d^3x\, \psi^* \frac{\vec{p}}{m} \psi$$

$$= \frac{\langle \vec{p} \rangle_t}{m} .$$

real

Let $\psi(x,t) = \sqrt{\rho(x,t)}\; e^{i\frac{S(x,t)}{\hbar}}$ This is always possible.

real & positive

Then

$$\psi^* \nabla \psi = e^{-i\frac{S}{\hbar}} \sqrt{\rho}\; \vec{\nabla} \sqrt{\rho}\, e^{i\frac{S}{\hbar}}$$

$$= e^{-i\frac{S}{\hbar}} \sqrt{\rho} \left[e^{i\frac{S}{\hbar}} \nabla\sqrt{\rho} + \sqrt{\rho}\, \frac{i}{\hbar} \nabla S\, e^{iS/\hbar} \right]$$

$$= \underbrace{\sqrt{\rho}\, \vec{\nabla}\sqrt{\rho}}_{\text{real}} + \underbrace{\frac{i}{\hbar} \rho\, \vec{\nabla} S}_{\text{imag.}}$$

So

$$\vec{j} = \frac{\hbar}{m} \operatorname{Im} \psi^* \nabla \psi = \frac{\rho}{m} \vec{\nabla} S .$$

For a plane wave

$$\psi(x,t) = c\, e^{i\frac{p\cdot x}{\hbar} - i\frac{Et}{\hbar}} \qquad E = \frac{p^2}{2m}$$

$$S = p\cdot x - Et \qquad \vec{\nabla} S = \vec{p} \qquad \vec{j} = \frac{\rho}{m}\nabla S = \frac{\nabla S}{m}$$

Call $\vec{v} \equiv \frac{\nabla S}{m}$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot \rho v = 0 + 0 = 0.$$

Classical limit

$$\psi(x,t) = \underbrace{\sqrt{\rho(x,t)}}_{\text{pos}}\, e^{i \overbrace{S(x,t)}^{\text{real}}/\hbar}$$

$H\psi = i\hbar\, \partial\psi/\partial t$ gives

$$-\frac{\hbar^2}{2m}\left[\nabla^2\sqrt{\rho} + \frac{2i}{\hbar}\vec{\nabla}\sqrt{\rho}\cdot\vec{\nabla}S - \frac{1}{\hbar^2}\sqrt{\rho}\,\vec{\nabla}S^2 + \frac{i}{\hbar}\sqrt{\rho}\,\nabla^2 S\right]$$

$$+\sqrt{\rho}\, V = i\hbar\left[\frac{\partial\sqrt{\rho}}{\partial t} + \frac{i}{\hbar}\sqrt{\rho}\,\frac{\partial S}{\partial t}\right].$$

dropping terms $O(\hbar)$ and $O(\hbar^2)$, assuming $\hbar|\nabla^2 S| << |\vec{\nabla} S|^2$

$$\frac{1}{2m}\sqrt{\rho}\,\vec{\nabla}S^2 + \sqrt{\rho}\, V = -\sqrt{\rho}\,\frac{\partial S}{\partial t} \quad \text{or}$$

$$\frac{1}{2m}(\vec{\nabla}S)^2 + V(x) + \frac{\partial S(x,t)}{\partial t} = 0 \qquad \text{H-J.} \qquad 1836$$

Stationary state case
Classically for a constant classical H, $S(x,t) = W(x) - Et$

$W(x)$ is H's characteristic function

$$\vec{p}_{class} = \vec{\nabla} S = \vec{\nabla} W$$

But back in 1836 there was no t.
No idea particles waved.

10/18
0/1

WKB

For a stationary state of energy E

$$S(x,t) = W(x) - Et \quad \text{and so}$$

$$\frac{1}{2m}(\vec{\nabla} S)^2 + V(x) + \frac{\partial S}{\partial t} = 0 \quad \text{becomes}$$

$$\frac{1}{2m}(\vec{\nabla} W)^2 + V(x) = E$$

In one dimension this is

$$\left(\frac{dW}{dx}\right)^2 = 2m\,(E - V(x)) \quad \text{whence}$$

$$W(x) = \pm \int^x dx' \sqrt{2m(E - V(x'))} \quad \text{and}$$

$$S(x,t) = \pm \int^x dx' \sqrt{2m(E - V(x'))} - Et.$$

For a stationary state, all the t dependence is in $e^{-iEt/\hbar}$.

So

$$\dot{\rho} = 0.$$

Thus the exact formula

$$\vec{j} = \frac{\rho}{m}\vec{\nabla}S \xrightarrow{d=1} j = \frac{\rho}{m}\frac{dS}{dx} = \frac{\rho}{m}W'$$

But

$$0 = \dot{\rho} + \vec{\nabla}\cdot\vec{j} \xrightarrow{d=1} 0 = \dot{\rho} + j' = j' \text{ whence}$$

$$0 = (\rho W')' = \rho' W' + \rho W''$$

so

$$\rho W' = \text{constant}$$

$$\rho(x) = \frac{\text{constant}}{\pm\sqrt{2m(E-V(x))}}$$

or

$$\sqrt{\rho} = \frac{\text{constant}'}{(E-V(x))^{\frac{1}{4}}} \propto \frac{1}{\sqrt{v}}$$

Particle density $\rho \sim 1/|v|$ is least where speed is greatest. classical.

So the JWKB solution is

$$\psi(x,t) \approx \frac{\text{constant}}{[E-V(x)]^{\frac{1}{4}}} e^{\left\{\pm\frac{i}{\hbar}\int^x dx'\sqrt{2m[E-V(x')]} - \frac{iEt}{\hbar}\right\}}$$

This is good when the terms we dropped are small, i.e., when $\hbar|\nabla^2 S| << (\vec{\nabla}S)^2$, etc.

This is a short-wavelength limit.

for $d=1$, it is $\quad \hbar |W''| << (W')^2$

Now

$$\lambda = \frac{\hbar}{p} = \frac{\hbar}{\sqrt{2m(E-V(x))}}$$

and

$$W = \pm \int^x dx' \sqrt{2m(E-V(x'))} \qquad \text{so}$$

$$W' = \pm \sqrt{2m(E-V(x))} \qquad \text{so}$$

$$W'' = \pm \frac{1}{2} \frac{(-2m\, V'(x))}{\sqrt{2m(E-V(x))}} = \mp \frac{m\, V'}{\sqrt{2m(E-V)}}$$

So the condition is

$$\frac{\hbar \; |V'|}{\sqrt{2m(E-V)}} << 2 \; (E-V) \qquad \text{or}$$

$$\lambda |V'| \approx |dV| << 2(E-V).$$

So in λ, dV should be small compared to $2(E-V)$. $\lambda |V'|/2KE << 1$.

when $V(x') > E$, we get

$$\psi(x,t) \approx \frac{\text{constant}}{(V(x)-E)^{\frac{1}{4}}} e^{\left(\pm \frac{1}{\hbar} \int^x dx' \sqrt{2m[V(x')-E]} - i\frac{Et}{\hbar}\right)}$$

where $V(x) = E$, neither solution is valid.

The fix is:

1) approximate $V(x)$ near x_0 by $V(x_0) + (x-x_0)V'(x_0)$.

2) Solve

$$-\frac{\hbar^2}{2m} u''(x) + V_0 + (x-x_0)V'(x_0)\, u(x) = E\, u(x)$$

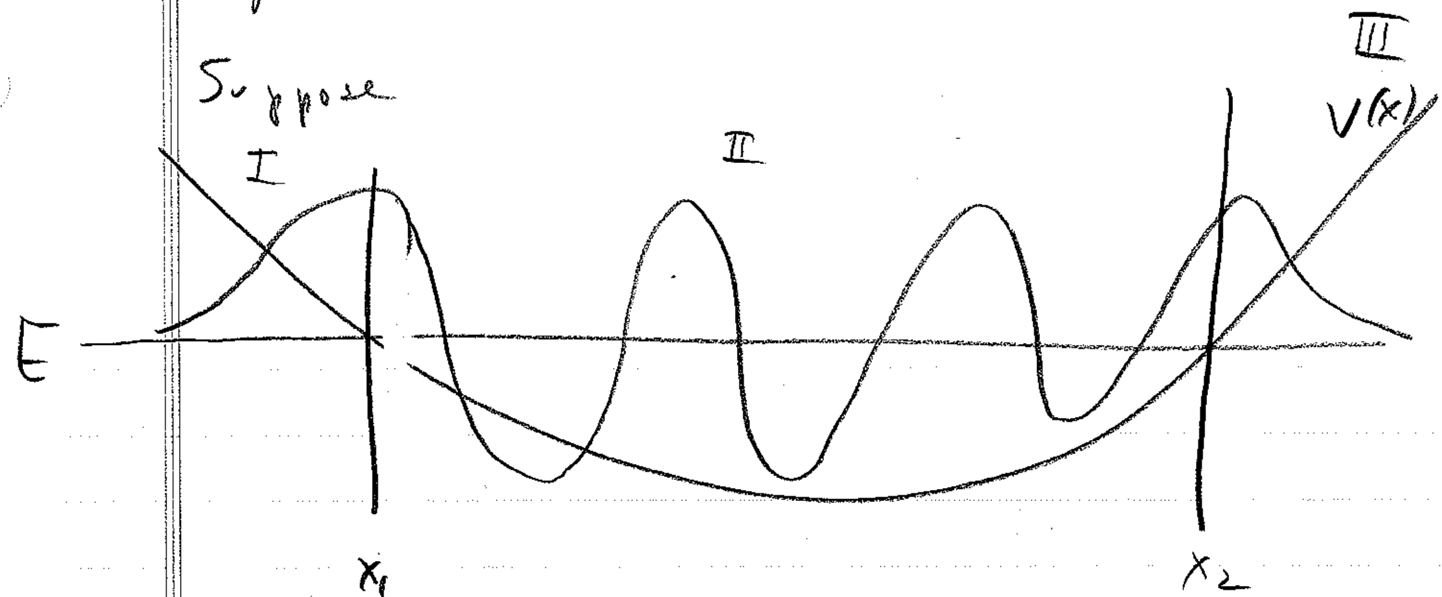
or

$$u''(x) = \frac{2m}{\hbar^2} V'(x_0)\,(x-x_0)\,u(x)$$

in terms of a Bessel function of order $\pm\frac{1}{3}$ near x_0.

3) Match this solution to the other two by adjusting the constants of integration.

Suppose



In region I

$$\frac{1}{(V(x)-E)^{\frac{1}{4}}} \exp\left[-\frac{1}{\hbar}\int_x^{x_1} dx' \sqrt{2m(V(x')-E)}\right] \quad \text{matches}$$

$$\frac{2}{(E-V(x))^{\frac{1}{4}}} \cos\left[\frac{1}{\hbar}\int_{x_1}^{x} dx' \sqrt{2m[E-V(x')]} - \frac{\pi}{4}\right]$$

in II

And in III

$$\mathrm{III} = \frac{1}{[V(x)-E]^{\frac{1}{4}}} \exp\left[-\frac{1}{\hbar}\int_{x_2}^{x} dx' \sqrt{2m[V(x')-E]}\right] \text{ matches}$$

$$\mathrm{II}_3 = \frac{2}{[E-V(x)]^{\frac{1}{4}}} \cos\left[-\frac{1}{\hbar}\int_{x}^{x_2} dx' \sqrt{2m[E-V(x')]} + \frac{\pi}{4}\right]$$

in II

The functions in II agree if

$$\frac{1}{\hbar}\int_{x_1}^{x}(\quad) - \frac{\pi}{4} = -\frac{1}{\hbar}\int_{x}^{x_2}(\quad) + \frac{\pi}{4} \quad \text{or}$$

$$\int_{x_1}^{x_2} dx' \sqrt{2m[E-V(x')]} = \frac{\pi}{2}\hbar + n\pi\hbar \quad \text{or}$$

the $n\pi$ is because one could apply (-1) to II_3 and III.

$$\oint dx' \sqrt{2m[E-V(x')]} = (2n+1)\pi\hbar = (n+\tfrac{1}{2})h$$

which is to be compared with the A. Sommerfeld W. Wilson condition

$$\oint p\,dq = nh \qquad 1915.$$

$$V(x) = \begin{cases} mgx & x>0 \\ \infty & x<0 \end{cases}$$

has solutions that are the same as the odd-parity solutions of

$$V(x) = mg|x|$$

which has $x_1 = -\frac{E}{mg}$ $x_2 = \frac{E}{mg}$. Then

$$\int_{-E/mg}^{E/mg} dx\,\sqrt{2m(E - mg|x|)} = (n_{odd} + \tfrac{1}{2})\pi\hbar \quad \text{or}$$

$$\int_0^{E/mg} dx\,\sqrt{2m(E-mgx)} = \tfrac{1}{2}(2n'+1+\tfrac{1}{2})\pi\hbar \qquad n' = 0,1,2,3$$

$$= (n' + \tfrac{3}{4})\pi\hbar = (n - \tfrac{1}{4})\pi\hbar$$

$$n = 1, 2, 3 \ldots$$

$$\frac{2}{3(2m^2 g)}\left[2m(E-mgx)\right]^{\frac{3}{2}}\Bigg]_{\frac{E}{mg}}^{0} = (n - \tfrac{1}{4})\pi\hbar$$

$$(2mE)^{3/2} = 3(n-\tfrac{1}{4})\pi\hbar\, m^2 g$$

$$E_n = \frac{\left(3(n-\frac{1}{4})\pi\hbar m^2 g\right)^{2/3}}{2m}$$

$$E_n = \tfrac{1}{2}\left[3(n-\tfrac{1}{4})\pi\right]^{2/3}(mg^2\hbar^2)^{\frac{1}{3}} \qquad n = 1,2,3\ldots$$

exact is $E_n = \frac{\lambda_n}{2^{1/3}}(mg^2\hbar^2)^{\frac{1}{3}}$ where $Ai(-\lambda_n) = 0$.
Better agreement at higher n. But good at $n = 1$ too.

$q\bar{q}$ potential may be

$$V(r) = \sigma r$$

with $\sigma = 1 \frac{GeV}{fm} \simeq 1.6 \times 10^{5}\ N \simeq 16$ tons.

Gravity gives $0.98 N$ on ball of 0.1 kg.

Propagators

$$K(x',t';x,t) = \langle x'| e^{-i\frac{H}{\hbar}(t'-t)} |x\rangle$$

If $\quad H|a'\rangle = E_{a'}|a'\rangle$, then

$$K(x',t';x,t) = \langle x'| \sum |a'\rangle\langle a'| e^{-i\frac{H}{\hbar}(t'-t)} |x\rangle$$

$$= \sum_{a'} \langle x'|a'\rangle e^{-i\frac{E_{a'}}{\hbar}(t'-t)} \langle a'|x\rangle.$$

Note that $K(x',t;x,t) = \delta^3(\vec{x}'-\vec{x})$.

If we add the boundary condition, reducing paus

$$K(x',t';x,t) = \langle x'| e^{-iH(t'-t)/\hbar} |x\rangle\, \theta(t'-t),$$

then

$$\left[-\frac{\hbar^2}{2m}\Delta' + V(x')\right] K(x't';xt)$$

$$= \sum_{a'} \langle x'|H|a'\rangle e^{-iE_{a'}(t'-t)/\hbar} \langle a'|x\rangle\, \theta(t'-t)$$

$$= \sum_{a'} \langle x'|a'\rangle E_{a'} e^{-iE_{a'}(t'-t)/\hbar} \langle a'|x\rangle\, \theta(t'-t)$$

while

$$-i\hbar\frac{\partial}{\partial t'} K = \sum \langle x'|a'\rangle(-E_{a'}) e^{-iE_{a'}(t'-t)/\hbar} \langle a'|x\rangle$$

$$-i\hbar\sum_{a'} \langle x'|a'\rangle e^{-i\frac{E_{a'}}{\hbar}(t'-t)} \langle a'|x\rangle\, \delta(t'-t)$$

So

$$\left[-\frac{\hbar^2}{2m}\Delta' + V(x') - i\hbar\frac{\partial}{\partial t'}\right] K(x't';xt) = -i\hbar\,\delta^3(x'-x)\,\delta(t'-t).$$