The Lorentz Group and the Dirac Matrices

A Lorentz transformation turns

\[ x'^\mu = \Lambda^\mu_\nu x^\nu \]

into

\[ x'^\mu = \Lambda^\mu_\nu x^\nu = \sum_{\nu=0}^{3} \Lambda^\mu_\nu x^\nu \]

(where the repeated index \( \nu \) is summed from 0 to 3) while preserving the Minkowski dot product

\[ x' \cdot y' = x' \cdot y - x^0 y^0 = \overrightarrow{x'} \cdot \overrightarrow{y} - x y^0 \]

in which \( y' \) is related to \( y \) by

\[ y'^\mu = \Lambda^\mu_\nu y^\nu. \]

Now \( x' \cdot y' \) is

\[ x' \cdot y' = x'^\mu y'^\nu \eta_{\mu \nu} \]

where

\[ \eta_{\mu \nu} = \eta^{\mu \nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]
So to satisfy (3) the Lorentz matrix \( \Lambda \) must obey the constraint
\[
x^\alpha y^\beta \eta_{\alpha \beta} = x^a y^b \eta_{ab}
= \Lambda^a_{\mu} x^\mu \Lambda^b_{\nu} y^\nu \eta_{ab},
\]
which holds as long as
\[
\eta_{\alpha \beta} = \Lambda^a_{\mu} \Lambda^b_{\nu} \eta_{ab}.
\]

These matrices \( \Lambda \) form a group because if \( \Lambda_1 \) and \( \Lambda_2 \) preserve all 4-dot products, then so will their product
\[
\Lambda_{12}^a_{\mu} = \Lambda_1^a_{\mu} \Lambda_2^c_{\nu} \eta_{\mu \nu} \eta_{\alpha \beta}
\]
preserve all 4-dot products
\[
x^{\prime \mu} = \Lambda_{12}^{\mu} x^\mu
\]
\[
y^{\prime \mu} = \Lambda_{12}^{\mu} y^\mu
\]
\[
x^{\prime \alpha} x^{\prime \beta} \eta_{\alpha \beta} = x^m y^n \eta_{mn}
= -x^0 y^0 - x^0 y^0 = -x \cdot y - x^0 y^0.
\]
These matrices form a group with identity element
\[ \Lambda^m_n = \delta^m_n \]
and inverse elements being those that reverse others. That is, if
\[ x'{}^n = \Lambda^m_n x^n \]
then \( \Lambda^{-1} \) is the Lorentz transformation that takes \( x^n \) back to \( x'{}^n \)
\[ x^n = (\Lambda^{-1})^m_n x'{}^n \]
This group of Lorentz transformations is the homogeneous Lorentz group. If we add in the translations so that
\[ x'{}^n = \Lambda^m_n x^n + a^m \]
then these transformations form the inhomogeneous Lorentz group (aka the Poincaré group).
A representation of the homogeneous Lorentz group is a set of matrices satisfying the multiplication law
\[ D(\Lambda_2) D(\Lambda_1) = D(\Lambda_2 \Lambda_1) \]  
(16)
of the homogeneous Lorentz group.

A transformation \( \Lambda \) of the form
\[ \Lambda^{\mu \nu} = \delta^{\mu \nu} + w^{\mu \nu}, \]  
(17)
where the \( w^{\mu \nu} \) are tiny. Neglecting terms like \( w^2 \), we see that this \( \Lambda \) will satisfy the key condition (5) if
\[ \eta_{\mu \nu} = (\delta^a \mu + w^a \mu)(\delta^b \nu + w^b \nu) \eta_{ab} \]
\[ = \delta^a \mu \delta^b \nu \eta_{ab} + w^a \mu \delta^b \nu \eta_{ab} \]
\[ + \delta^a \mu w^b \nu \eta_{ab} \]
\[ = \eta_{\mu \nu} + w^a \mu \eta_{a \nu} + w^b \nu \eta_{b \mu}. \]  
(18)
Expressions like \( w^a \mu \eta_{a \nu} \) occur so often that they have a special notation
\[ w^a \mu \eta_{a \nu} = w_{\nu \mu}, \]  
(19)
Since \( \eta_{\mu \nu} \) is by (6) the same as the identity matrix except for a \(-1\) instead of a \(+1\) in the 00 spot

\[
W_{\mu \nu} = W^\nu_\lambda \quad \text{if } \nu < 1, 2, 3 \quad (20)
\]

while

\[
W_{\mu \nu} = -W^\nu_\mu \quad \text{if } \nu < 0. \quad (21)
\]

So \( \Lambda^\mu_\nu = \delta^\mu_\nu + W_{\mu \nu} \) is a Lorentz transformation.

\[
\eta_{\mu \nu} = \delta_{\mu \nu} + W_{\mu \nu} + \Lambda^\mu_\nu \quad (22)
\]

that is, if \( w \) is antisymmetric,

\[
W_{\mu \nu} = -W_{\nu \mu}. \quad (23)
\]

How do derivatives transform? If

\[
x'^\mu = \Lambda^\mu_\nu x^\nu \quad \text{then} \quad x^\nu = \Lambda^{-1\nu}_\mu x'^\mu \quad (24)
\]

and so by the chain rule

\[
\frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} = \Lambda^{-1\nu}_\mu \frac{\partial}{\partial x^\nu}. \quad (25)
\]
We may greatly simplify some of what follows by setting both \( t \) and \( c \) equal to unity. We then can write the unitary operator \( U(1+w) \) that represents the \( U \) \( (1+w) \) transformation

\[
\Lambda^m \nu = f^m \nu + \omega^m \nu \quad (26)
\]
as

\[
U(1+w) = e^{\frac{i}{2} \omega_{\mu \nu} J^{\mu \nu}} \approx 1 + \frac{i}{2} \omega_{mn} J^{mn} \quad (27)
\]

Since \( \omega \) is anti-symmetric, the Lorentz generators \( J^{\mu \nu} \) must also be anti-symmetric

\[
J^{\mu \nu} = -J^{\nu \mu} \quad (28)
\]

For any symmetric piece of \( J \) would cancel in (27). By considering the product

\[
U(\Lambda) U(1+w) U^{-1}(\Lambda) = U(\Lambda (1+w) \Lambda^{-1}) \quad (29)
\]

one can show that

\[
U(\Lambda) J^{\rho \sigma} U^{-1}(\Lambda) = \Lambda^{\rho \nu} \Lambda^{\sigma \mu} J^{\mu \nu} \quad (30)
\]

By now letting \( \Lambda \) itself be true, one
for the may show that
\[ i \left[ \frac{1}{2} \omega_{\mu \nu}, J^{\mu \nu} \right] = \omega_{\mu} P \sigma^{\nu} + \omega_{\nu} P \sigma^{\mu}, \]
and so that the generators $J^{\mu \nu}$ of Lorentz transformations satisfy
\[ i \left[ J^{\mu \nu}, J^{\rho \sigma} \right] = \eta^{\mu \rho} J^{\nu \sigma} - \eta^{\mu \sigma} J^{\nu \rho} + \eta^{\nu \sigma} J^{\rho \mu} + \eta^{\nu \rho} J^{\sigma \mu}. \]

Similar work on the inhomogeneous Lorentz group with generators $P^\mu$ and $P^\nu$ shows that
\[ i \left[ P^\mu, J^{\rho \sigma} \right] = \eta^{\mu \rho} P^{\sigma} - \eta^{\mu \sigma} P^{\rho}, \]
and
\[ \left[ P^\mu, P^\nu \right] = 0. \]

In all these equations, the summation convention
\[ \sum_{a = 0}^{3} X_a Y^a = \frac{1}{2} \sum_{a = 0}^{3} X_a Y^a \]
is used.

So we may find a representation of the homogeneous Lorentz group if
we can find matrices $J^{\mu\nu}$ that are anti-symmetric

$$J^{\mu\nu} = -J^{\nu\mu}$$

and that satisfy (3.2)

$$i \left[ J^{\mu\nu}, J^{\rho\sigma} \right] = \eta^{\mu\rho} J^{\sigma\nu} - \eta^{\nu\rho} J^{\mu\sigma} - \eta^{\sigma\tau} J^{\nu\mu} + \eta^{\tau\nu} J^{\sigma\mu}.$$

We need only set

$$D(1) = e^{i \frac{\omega}{2} J^{\mu\nu}}$$

or

$$D(1 + i \omega) = 1 + i \frac{\omega}{2} J^{\mu\nu}$$

if $\omega$ is tiny.

To find the matrices $J^{\mu\nu}$ which represent the generators $J^{\mu\nu}$ we introduce four $4 \times 4$ matrices — the Dirac matrices — that satisfy the anti-commutation relation

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = \{\gamma^\mu, \gamma^\nu\} = \sum \gamma^\mu \gamma^\nu + \sum \gamma^\nu \gamma^\mu = 2 \gamma^\mu.$$
One may verify that the matrices

$$\gamma^0 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^\nu = -i \begin{pmatrix} 0 & \sigma^\nu \\ -\sigma^\nu & 0 \end{pmatrix}$$

in which $1$ is the $2\times2$ identity matrix and the $\sigma^\nu$ are the $3$ Pauli matrices. For instance,

$$\gamma' = -i \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix} \quad \gamma^2 = -i \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}$$

and so $\gamma'$ and $\gamma^2$ anti-commute

$$\gamma' \gamma^2 = - \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} = - \begin{pmatrix} 0 & \sigma^1 \sigma^2 \\ -\sigma^1 \sigma^2 & 0 \end{pmatrix}$$

Since

$$\sigma^i \sigma^j = \delta^i_j + \sum_{k=1}^3 i \epsilon^{ijk} \sigma^k$$

$\sigma^1 \sigma^2 = i \sigma^3$ and so

$$\gamma' \gamma^2 = -i \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}$$
\[ Y^2 Y' = \begin{pmatrix} \sigma_2 \sigma_1 & 0 \\ 0 & 0 \end{pmatrix} = -i \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \]

and so
\[ \{Y', Y^2\} = 2 \eta^{12} = 0. \tag{46} \]

Also,
\[ \{Y^0, Y^a\} = 2 Y^0 \tau^a = -2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2 \eta^{00}. \tag{47} \]

Any set of 4 \(4 \times 4\) matrices \(Y^a\) that satisfy
\[ \{Y^0, Y^a\} = 2 \eta^{0a} \tag{48} \]

are called Dirac matrices; any set of Dirac matrices suffices to form a representation of the Lorentz group.

Note that if \( \Sigma \) is any \(4 \times 4\) non-singular matrix (i.e., matrix with an inverse) then if the \( Y^a \)'s obey (40) then
So these are infinitely many sets of divisible numbers.  

\[ S(x_1, x_2, y_1, y_2) = 2y_1y_2 \]  

\[ S(y_1, y_2, x_1, x_2) = 2y_1y_2 \]  

and so on.

Theorem: \( Y \) is defined by \( Y = \frac{1}{2} (x_1 + y_1) \) because \( y \) also obey (40).

\[ Y = 4m \]  

\[ S = 8 \]  

\[ x_1, y_1 \mid Y \]  

\[ x_2, y_2 \mid Y \]  

\[ 5 \]
\[ J^\nu = - J^\nu \]

and they satisfy

\[ \left[ J^\nu, \gamma^\rho \right] = -i \gamma^\nu \gamma^\rho + i \gamma^\rho \gamma^\nu \]  \hspace{1cm} (66)

One may now use this relation (56) to show that the \( J^\nu \)'s obey the commutation relation (37) and so furnish a \( 4 \times 4 \) representation of the Lorentz group

\[ D(1 + \omega) = 1 + \frac{i}{2} \omega_{\mu\nu} J^{\mu\nu} \]  \hspace{1cm} (57)

Rule (56) implies that

\[ D(\Lambda) \gamma^\rho D^{-1}(\Lambda) = \Lambda_0^\rho \gamma^\sigma \]  \hspace{1cm} (58)

which means that in the representation \( D(\Lambda) \) the Dirac matrices transform as a vector. Obviously

\[ D(\Lambda) D^{-1}(\Lambda) = 1 \]  \hspace{1cm} (59)

and so the unit matrix is a scalar.