Dirac's Delta Function

In one dimension, $\delta(x)$ is defined by

$$f(x) = \int_{-\infty}^{\infty} dy\ \delta(x-y)\, f(y) \qquad \text{(def.)}$$

for all suitably smooth functions $f(x)$.

If f and g are related by Fourier's transform

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}}\, e^{ikx}\, g(k)$$

$$g(k) = \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi}}\, e^{-iky}\, f(y)$$

then

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} dy\ e^{ik(x-y)}\, f(y)$$

$$= \int_{-\infty}^{\infty} dy \left[\int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ik(x-y)} \right] f(y) .$$

The definition (def) tells us that

$$\delta(x-y) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ik(x-y)} .$$

Since δ is real, we also have

$$\delta(x-y) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{-ik(x-y)}.$$

In 3 dimensions,

$$\delta^{(3)}(\vec{x}-\vec{y}) = \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3}\, e^{\pm i\vec{k}\cdot(\vec{x}-\vec{y})}.$$

We want (in 1 dimension)

$$\int_{-\infty}^{\infty} dp\, |p\rangle\langle p| = \int_{-\infty}^{\infty} dx\, |x\rangle\langle x| = I,$$

$$\langle x' | x\rangle = \delta(x-x'),$$

and

$$\langle p' | p\rangle = \delta(p-p').$$

And we know that $\langle x|p\rangle = e^{ixp/\hbar} N$ where N is a normalization constant. To find it, we set

$$\langle x'|x\rangle = \int_{-\infty}^{\infty} dp\, \langle x'|p\rangle\langle p|x\rangle = N^2 \int_{-\infty}^{\infty} dp\, e^{i(x'-x)p/\hbar}$$

$$= \delta(x'-x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{i(x'-x)k} = \int \frac{dp}{2\pi\hbar}\, e^{i(x'-x)p/\hbar}.$$

So $N^2 = \frac{1}{2\pi\hbar} = \frac{1}{h}$ and $N = \frac{1}{\sqrt{h}}$.