18. Consider an orbital angular-momentum eigenstate $|l=2, m=0\rangle$. Suppose this state is rotated by an angle β about the y-axis. Find the probability for the new state to be found in $m = 0$, ± 1, and ± 2. (The spherical harmonics for $l = 0$, 1, and 2 given in Appendix A may be useful.)

3.18 We want to compute the amplitudes

$$\langle 2, m | e^{-i\beta J_2/\hbar} | 2, 0\rangle = \int d\Omega \langle 2, m | \theta, \phi \rangle\langle \theta, \phi | e^{-i\beta J_2/\hbar} | 2, 0\rangle$$

$$= \int d\Omega \langle 2, m | \theta, \phi \rangle\langle \theta', \phi' | 2, 0\rangle$$

$$= \int d\Omega \, Y_2^{m*}(\theta, \phi) \, Y_2^0(\theta', \phi')$$

$$= \int d\Omega \, Y_2^{m*}(\theta, \phi) \sqrt{\frac{5}{16\pi}} \, (3\cos^2\theta' - 1) \, .$$

$|\theta', \phi'\rangle = \exp(i\beta J_2/\hbar) \, |\theta, \phi\rangle$. We use the web hint

In problem 3.18, you might use the expression I derived in class for the 3×3 real orthogonal matrix that represents a right-handed rotation by $\theta = |\vec{\theta}|$ radians about the axis $\hat{\theta} = \vec{\theta}/\theta$:

$$e^{-i\vec{\theta}\cdot\vec{J}/\hbar} = \cos\theta \, I - i\hat{\theta}\cdot\vec{J}\sin\theta + (1-\cos\theta)\,\hat{\theta}(\hat{\theta})^{\mathsf{T}}$$

in which the generators $(J_k)_{ij} = i\hbar\epsilon_{ikj}$ satisfy $[J_i, J_j] = i\hbar\epsilon_{ijk}J_k$ and T means transpose. In terms of indices, this formula for $R(\vec{\theta}) = e^{-i\vec{\theta}\cdot\vec{J}/\hbar}$ is

$$R(\vec{\theta})_{ij} = \delta_{ij}\cos\theta - \sin\theta\,\epsilon_{ijk}\hat{\theta}_k + (1-\cos\theta)\,\hat{\theta}_i\hat{\theta}_j.$$

In matrix form $R(\vec{\theta})$ is:

$$\begin{pmatrix} \cos\theta + \hat{\theta}_1^2(1-\cos\theta) & -\hat{\theta}_3\sin\theta + \hat{\theta}_1\hat{\theta}_2(1-\cos\theta) & \hat{\theta}_2\sin\theta + \hat{\theta}_1\hat{\theta}_3(1-\cos\theta) \\ \hat{\theta}_3\sin\theta + \hat{\theta}_2\hat{\theta}_1(1-\cos\theta) & \cos\theta + \hat{\theta}_2^2(1-\cos\theta) & -\hat{\theta}_1\sin\theta + \hat{\theta}_2\hat{\theta}_3(1-\cos\theta) \\ -\hat{\theta}_2\sin\theta + \hat{\theta}_3\hat{\theta}_1(1-\cos\theta) & \hat{\theta}_1\sin\theta + \hat{\theta}_3\hat{\theta}_2(1-\cos\theta) & \cos\theta + \hat{\theta}_3^2(1-\cos\theta) \end{pmatrix}.$$