Complement D_{VI}

ANGULAR MOMENTUM OF STATIONARY STATES OF A TWO-DIMENSIONAL HARMONIC OSCILLATOR

1. Introduction
   a. *Review of the classical problem*
   b. *The problem in quantum mechanics*
2. Classification of the stationary states by the quantum numbers n_x and n_y
   a. *Energies; stationary states*
   b. *H_{xy} does not constitute a C.S.C.O. in $\mathscr{E}_{xy}$*
3. Classification of the stationary states in terms of their angular momenta
   a. *Significance and properties of the operator L_z*
   b. *Right and left circular quanta*
   c. *Stationary states of well-defined angular momentum*
   d. *Wave functions associated with the eigenstates common to H_{xy} and L_z*
4. Quasi-classical states
   a. *Definition of the states $| \alpha_x, \alpha_y \rangle$ and $| \alpha_d, \alpha_g \rangle$*
   b. *Mean values and root-mean-square deviations of the various observables*

In this complement, we shall be concerned with the quantum mechanical properties of a two-dimensional harmonic oscillator. The quantum mechanical problem is exactly soluble and does not involve complicated calculations. Furthermore, this subject provides an opportunity to study a simple application of the properties of the orbital angular momentum **L**, since, as we shall see, the stationary states of such an oscillator can be classified with respect to the possible values of the observable L_z. In addition, the results obtained will be useful in the next complement, E_{VI}.

1. Introduction

a. REVIEW OF THE CLASSICAL PROBLEM

A physical particle always moves in three-dimensional space. However, if its potential energy depends only on x and y, the problem can be treated in two dimensions. We shall assume here that this potential energy can be written:

$$V(x, y) = \frac{\mu}{2} \omega^2 (x^2 + y^2) \tag{1}$$

where μ is the mass of the particle and ω is a constant. The classical Hamiltonian of the system is then:

$$\mathscr{H} = \mathscr{H}_{xy} + \mathscr{H}_z \tag{2}$$

with:

$$\mathscr{H}_{xy} = \frac{1}{2\mu}(p_x^2 + p_y^2) + \frac{1}{2}\mu\omega^2(x^2 + y^2)$$

$$\mathscr{H}_z = \frac{1}{2\mu}p_z^2 \tag{3}$$

where p_x, p_y, p_z are the three components of the momentum $\mathbf{p}$ of the particle. $\mathscr{H}_{xy}$ is a two-dimensional harmonic oscillator Hamiltonian.

The equations of motion can easily be integrated to yield:

$$\begin{cases} p_z(t) = p_0 \\ z(t) = \dfrac{p_0}{\mu}t + z_0 \end{cases} \tag{4}$$

$$\begin{cases} x(t) = x_M \cos(\omega t - \varphi_x) \\ p_x(t) = -\mu\omega x_M \sin(\omega t - \varphi_x) \end{cases} \tag{5}$$

$$\begin{cases} y(t) = y_M \cos(\omega t - \varphi_y) \\ p_y(t) = -\mu\omega y_M \sin(\omega t - \varphi_y) \end{cases} \tag{6}$$

where $p_0, z_0, x_M, \varphi_x, y_M, \varphi_y$ are constants which depend on the initial conditions (we assume x_M and y_M to be positive).

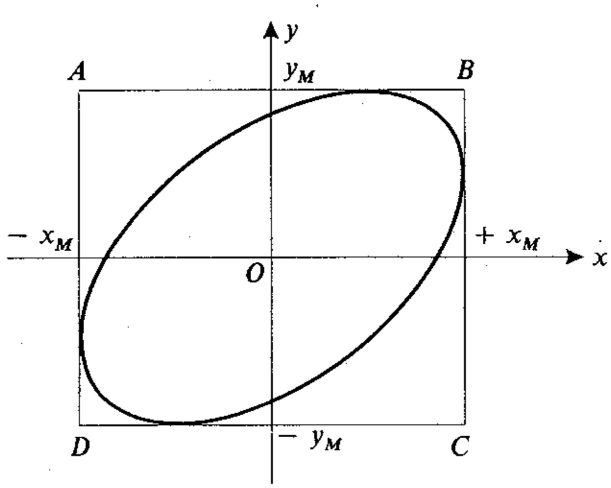

FIGURE 1

Projection of the classical trajectory of a particle in a two-dimensional harmonic potential onto the xOy plane; we obtain an ellipse inscribed in the rectangle $ABCD$.

We see that the projection of the particle onto Oz describes a uniform motion with a velocity of p_0/μ. The projection onto the xOy plane describes an ellipse inscribed in the rectangle $ABCD$ of figure 1. The direction the particle takes on this ellipse depends on the phase difference $\varphi_y - \varphi_x$. When $\varphi_y - \varphi_x = \pm\pi$, the ellipse reduces to the line AC. When $\varphi_y - \varphi_x$ is between $-\pi$ and 0, the particle moves clockwise on the ellipse ("left-handed" motion), with the axes of the ellipse parallel to Ox and Oy for $\varphi_y - \varphi_x = -\pi/2$. When $\varphi_y - \varphi_x = 0$, the ellipse reduces to the line BD. Finally, when $\varphi_y - \varphi_x$ is between 0 and π, the particle moves counterclockwise on the ellipse ("right-handed" motion), with the axes parallel to Ox and Oy for $\varphi_y - \varphi_x = +\pi/2$. Note that the ellipse reduces to a circle if $\varphi_y - \varphi_x = \pm\pi/2$ and $x_M = y_M$.

It is easy to determine several constants of the motion related to the projection of the motion onto the xOy plane:

– the total energy $\mathscr{H}_{xy}$, which, according to (3), (5), (6), is equal to:

$$\mathscr{H}_{xy} = \frac{1}{2}\mu\omega^2(x_M^2 + y_M^2) \tag{7}$$

– the energies:

$$\mathscr{H}_x = \frac{1}{2}\mu\omega^2 x_M^2 \tag{8-a}$$

$$\mathscr{H}_y = \frac{1}{2}\mu\omega^2 y_M^2 \tag{8-b}$$

of the projections of the motion onto Ox and Oy;

– the component of the orbital angular momentum $\mathscr{L}$ of the particle along Oz:

$$\mathscr{L}_z = xp_y - yp_x \tag{9}$$

which, according to (5) and (6), is equal to:

$$\mathscr{L}_z = \mu\omega x_M y_M \sin(\varphi_y - \varphi_x) \tag{10}$$

We see that $\mathscr{L}_z$ is positive or negative depending on whether the motion is counterclockwise ($0 < \varphi_y - \varphi_x < \pi$) or clockwise ($-\pi < \varphi_y - \varphi_x < 0$). $\mathscr{L}_z$ is zero for the two rectilinear motions ($\varphi_y - \varphi_x = \pm\pi$ and $\varphi_y - \varphi_x = 0$). Finally, for a motion at a given energy, that is, according to (7), for a fixed value of $x_M^2 + y_M^2$, $|\mathscr{L}_z|$ is maximal when $\varphi_y - \varphi_x = \pm\pi/2$ and the product $x_M y_M$ is maximal, which implies $x_M = y_M$. Of all motions at a given energy, it is the counterclockwise (clockwise) motion which corresponds to the maximal (minimal) algebraic value of $\mathscr{L}_z$.

b. THE PROBLEM IN QUANTUM MECHANICS

The quantization rules of chapter III enable us to obtain H, H_{xy}, H_z from $\mathscr{H}$, $\mathscr{H}_{xy}$, $\mathscr{H}_z$. The stationary states $|\varphi\rangle$ of the particle are given by:

$$H|\varphi\rangle = (H_{xy} + H_z)|\varphi\rangle = E|\varphi\rangle \tag{11}$$

with:

$$H_{xy} = \frac{P_x^2 + P_y^2}{2\mu} + \frac{1}{2}\mu\omega^2(X^2 + Y^2) \tag{12-a}$$

$$H_z = \frac{P_z^2}{2\mu} \tag{12-b}$$

According to the results of complement F_I, we know that we can choose a basis of eigenstates of H composed of vectors of the form:

$$| \varphi \rangle = | \varphi_{xy} \rangle \otimes | \varphi_z \rangle \tag{13}$$

where $| \varphi_{xy} \rangle$ is an eigenvector of H_{xy} in the state space $\mathscr{E}_{xy}$ associated with the variables x and y:

$$H_{xy} | \varphi_{xy} \rangle = E_{xy} | \varphi_{xy} \rangle \tag{14}$$

and $| \varphi_z \rangle$ is an eigenvector of H_z in the space $\mathscr{E}_z$ associated with the variable z:

$$H_z | \varphi_z \rangle = E_z | \varphi_z \rangle \tag{15}$$

The total energy associated with the state (13) is then:

$$E = E_{xy} + E_z \tag{16}$$

Now, equation (15), which in fact describes the stationary states of a free particle in a one-dimensional problem, can be solved immediately; it yields:

$$\langle z | \varphi_z \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ip_z z/\hbar} \tag{17}$$

(where p_z is an arbitrary real constant), with:

$$E_z = \frac{p_z^2}{2\mu} \tag{18}$$

The problem therefore reduces to the determination of the solutions of equation (14), that is, the energies and stationary states of a two-dimensional harmonic oscillator. This is the problem we shall now try to solve.

We shall see that the eigenvalues E_{xy} of H_{xy} are degenerate : H_{xy} alone does not constitute a C.S.C.O. in $\mathscr{E}_{xy}$. We must therefore add one or several other observables to H_{xy} in order to construct a C.S.C.O. In fact, we find in quantum mechanics the same constants of the motion as in classical mechanics: H_x and H_y, the energies of the projection of the motion onto Ox and Oy; and L_z, the component along Oz of the orbital angular momentum $\mathbf{L}$. Since L_z commutes with neither H_x nor H_y, we shall see that a C.S.C.O. can be formed of H_{xy}, H_x and H_y (§2) or of H_{xy} and L_z (§3).

COMMENTS:

(*i*) Formula (18) indicates that the eigenvalues E_z of H_z are all two-fold degenerate in the space $\mathscr{E}_z$. Furthermore, the degeneracy in $\mathscr{E} = \mathscr{E}_{xy} \otimes \mathscr{E}_z$ of the eigenvalues (16) of the total Hamiltonian H is not due solely to the degeneracy of E_{xy} in $\mathscr{E}_{xy}$ and of E_z in $\mathscr{E}_z$: two eigenvectors of H of the form (13) can have the same total energy E without their corresponding values of E_{xy} (and of E_z) being equal.

(*ii*) H commutes with the component L_z of $\mathbf{L}$, but not with L_x and L_y. This results from the fact that the potential energy written in (1) is rotation-invariant only about Oz. Moreover, of the three operators L_x, L_y and L_z, only one, L_z, acts only in $\mathscr{E}_{xy}$. In the study of the two-dimensional harmonic oscillator, therefore, we shall use only the observable L_z. In complement B_{VII}, we shall study the isotropic three-dimensional harmonic oscillator, whose potential energy is invariant with respect to any rotation about an axis which passes through the origin; we shall see that all the components of **L** then commute with the Hamiltonian.

2. Classification of the stationary states by the quantum numbers n_x and n_y

a. ENERGIES; STATIONARY STATES

To obtain the solutions of the eigenvalue equation (14), note that H_{xy} can be written:

$$H_{xy} = H_x + H_y \tag{19}$$

where H_x and H_y are both Hamiltonians of one-dimensional harmonic oscillators:

$$\begin{aligned} H_x &= \frac{P_x^2}{2\mu} + \frac{1}{2}\mu\omega^2 X^2 \\ H_y &= \frac{P_y^2}{2\mu} + \frac{1}{2}\mu\omega^2 Y^2 \end{aligned} \tag{20}$$

We know the eigenstates $|\varphi_{n_x}\rangle$ of H_x in $\mathscr{E}_x$ and the eigenstates $|\varphi_{n_y}\rangle$ of H_y in $\mathscr{E}_y$. Their energies are, respectively, $E_x = (n_x + 1/2)\hbar\omega$ and $E_y = (n_y + 1/2)\hbar\omega$ (where n_x and n_y are positive integers or zero). The eigenstates of H_{xy} can thus be chosen in the form:

$$|\varphi_{n_x,n_y}\rangle = |\varphi_{n_x}\rangle \otimes |\varphi_{n_y}\rangle \tag{21}$$

where the corresponding energy E_{xy} is given by:

$$\begin{aligned} E_{xy} &= \left(n_x + \frac{1}{2}\right)\hbar\omega + \left(n_y + \frac{1}{2}\right)\hbar\omega \\ &= (n_x + n_y + 1)\hbar\omega \end{aligned} \tag{22}$$

According to the properties of the one-dimensional harmonic oscillator, E_x is non-degenerate in $\mathscr{E}_x$, and E_y in $\mathscr{E}_y$. Consequently, a vector $|\varphi_{n_x,n_y}\rangle$ of $\mathscr{E}_{xy}$ which is unique to within a constant factor corresponds to a pair $\{n_x, n_y\}$: H_x and H_y form a C.S.C.O. in $\mathscr{E}_{xy}$.

It will prove convenient to use the operators a_x and a_y (destruction operators of a quantum, relative to Ox and Oy respectively), defined by:

$$\begin{aligned} a_x &= \frac{1}{\sqrt{2}}\left(\beta X + i\frac{P_x}{\beta\hbar}\right) \\ a_y &= \frac{1}{\sqrt{2}}\left(\beta Y + i\frac{P_y}{\beta\hbar}\right) \end{aligned} \tag{23}$$

with :

$$\beta = \sqrt{\frac{\mu\omega}{\hbar}} \tag{24}$$

Since a_x and a_y act in different spaces, $\mathscr{E}_x$ and $\mathscr{E}_y$, the only non-zero commutators between the four operators a_x, a_y, $a_x^\dagger$, $a_y^\dagger$, are:

$$[a_x, a_x^\dagger] = [a_y, a_y^\dagger] = 1 \tag{25}$$

The operators N_x (the number of quanta relative to the Ox axis) and N_y (the number of quanta relative to the Oy axis) are given by:

$$\begin{aligned} N_x &= a_x^\dagger a_x \\ N_y &= a_y^\dagger a_y \end{aligned} \tag{26}$$

which enables us to write H_{xy} in the form:

$$H_{xy} = H_x + H_y = (N_x + N_y + 1)\,\hbar\omega \tag{27}$$

We have, obviously:

$$\begin{aligned} N_x \,|\, \varphi_{n_x,n_y} \rangle &= n_x \,|\, \varphi_{n_x,n_y} \rangle \\ N_y \,|\, \varphi_{n_x,n_y} \rangle &= n_y \,|\, \varphi_{n_x,n_y} \rangle \end{aligned} \tag{28}$$

The ground state $|\varphi_{0,0}\rangle$ is given by:

$$|\, \varphi_{0,0} \rangle = |\, \varphi_{n_x=0} \rangle \otimes |\, \varphi_{n_y=0} \rangle \tag{29}$$

The state $|\varphi_{n_x,n_y}\rangle$ defined by (21) can be obtained from $|\varphi_{0,0}\rangle$ by the successive application of the operators $a_x^\dagger$ and $a_y^\dagger$:

$$|\, \varphi_{n_x,n_y} \rangle = \frac{1}{\sqrt{n_x!\, n_y!}} (a_x^\dagger)^{n_x} (a_y^\dagger)^{n_y} \,|\, \varphi_{0,0} \rangle \tag{30}$$

The corresponding wave function is the product of $\varphi_{n_x}(x)$ and $\varphi_{n_y}(y)$ [*cf.* complement B_V, formula (35)]:

$$\varphi_{n_x,n_y}(x, y) = \frac{\beta}{\sqrt{\pi (2)^{n_x+n_y} (n_x)!\,(n_y)!}} \, e^{-\beta^2(x^2+y^2)/2} H_{n_x}(\beta x)\, H_{n_y}(\beta y) \tag{31}$$

b. H_{xy} **DOES NOT CONSTITUE A C.S.C.O. IN** $\mathscr{E}_{xy}$

We see from (22) that the eigenvalues of H_{xy} are of the form:

$$E_{xy} = E_n = (n + 1)\,\hbar\omega \tag{32}$$

where:

$$n = n_x + n_y \tag{33}$$

is a positive integer or zero. To each value of the energy correspond the various orthogonal eigenvectors:

$$|\, \varphi_{n_x=n,n_y=0} \rangle, \,|\, \varphi_{n_x=n-1,n_y=1} \rangle, \ldots, |\, \varphi_{n_x=0,n_y=n} \rangle \tag{34}$$

Since there are $(n + 1)$ of these vectors, the eigenvalue E_n is $(n + 1)$-fold degenerate in $\mathscr{E}_{xy}$. H_{xy} alone does not, therefore, constitute a C.S.C.O. On the other hand, we have seen that $\{ H_x, H_y \}$ is a C.S.C.O.; this is also, obviously, true of $\{ H_{xy}, H_x \}$ and $\{ H_{xy}, H_y \}$.

3. Classification of the stationary states in terms of their angular momenta

a. SIGNIFICANCE AND PROPERTIES OF THE OPERATOR L_z

In the preceding section, we identified the stationary states by the quantum numbers n_x and n_y. But the Ox and Oy axes do not enjoy a privileged position in this problem. Since the potential energy is invariant under rotation about Oz, we could just as well have chosen another system of orthogonal axes Ox' and Oy' in the xOy plane; we would have then obtained stationary states different from the preceding ones.

Therefore, in order to take better advantage of the symmetry of the problem, we shall now consider the component L_z of the angular momentum, defined by:

$$L_z = XP_y - YP_x \tag{35}$$

Expressing X and P_x in terms of a_x and $a_x^\dagger$, and Y and P_y in terms of a_y and $a_y^\dagger$, we get:

$$L_z = i\hbar(a_x a_y^\dagger - a_x^\dagger a_y) \tag{36}$$

Now, the expression for H_{xy} in terms of the same operators is:

$$H_{xy} = (a_x^\dagger a_x + a_y^\dagger a_y + 1)\,\hbar\omega \tag{37}$$

Since:

$$\begin{aligned} \left[a_x a_y^\dagger, a_x^\dagger a_x + a_y^\dagger a_y\right] &= a_x a_y^\dagger - a_x a_y^\dagger = 0 \\ \left[a_x^\dagger a_y, a_x^\dagger a_x + a_y^\dagger a_y\right] &= -\, a_x^\dagger a_y + a_x^\dagger a_y = 0 \end{aligned} \tag{38}$$

we find that:

$$\left[H_{xy}, L_z\right] = 0 \tag{39}$$

We shall therefore look for a basis of eigenvectors common to H_{xy} and L_z.

b. RIGHT AND LEFT CIRCULAR QUANTA

We introduce the operators a_d and a_g defined by:

$$\begin{aligned} a_d &= \frac{1}{\sqrt{2}}(a_x - i a_y) \\ a_g &= \frac{1}{\sqrt{2}}(a_x + i a_y) \end{aligned} \tag{40}$$

We see from this definition that the action of a_d (or a_g) on $| \varphi_{n_x,n_y} \rangle$ yields a state which is a linear combination of $| \varphi_{n_x-1,n_y} \rangle$ and $| \varphi_{n_x,n_y-1} \rangle$, that is, a stationary state which has one less energy quantum $\hbar\omega$. Similarly, the action of $a_d^\dagger$ (or $a_g^\dagger$) on $| \varphi_{n_x,n_y} \rangle$ yields another stationary state which has one more energy quantum. In fact, we shall see that a_d (or a_g) is analogous to a_x (or a_y), and that a_d and a_g can be interpreted as being destruction operators of a right and left "circular quantum" respectively.

First of all, using (40) and (25), it is simple to verify that the only non-zero commutators between the four operators a_d, a_g, $a_d^\dagger$, $a_g^\dagger$ are:

$$[a_d, a_d^\dagger] = [a_g, a_g^\dagger] = 1 \tag{41}$$

These relations are indeed analogous to (25). Moreover, H_{xy} can be written, in terms of these operators, in a way that is similar to (37); since:

$$\begin{aligned} a_d^\dagger a_d &= \frac{1}{2}(a_x^\dagger a_x + a_y^\dagger a_y - i a_x^\dagger a_y + i a_x a_y^\dagger) \\ a_g^\dagger a_g &= \frac{1}{2}(a_x^\dagger a_x + a_y^\dagger a_y + i a_x^\dagger a_y - i a_x a_y^\dagger) \end{aligned} \tag{42}$$

we have:

$$H_{xy} = (a_d^\dagger a_d + a_g^\dagger a_g + 1)\,\hbar\omega \tag{43}$$

In addition, using (36), we see that:

$$L_z = \hbar(a_d^\dagger a_d - a_g^\dagger a_g) \tag{44}$$

If we introduce the operators N_d and N_g (the number of right and left "circular quanta"):

$$\begin{aligned} N_d &= a_d^\dagger a_d \\ N_g &= a_g^\dagger a_g \end{aligned} \tag{45}$$

formulas (43) and (44) become:

$$\begin{aligned} H_{xy} &= (N_d + N_g + 1)\hbar\omega \\ L_z &= \hbar(N_d - N_g) \end{aligned} \tag{46}$$

Thus, while maintaining H in a form as simple as (27), we have simplified that of L_z.

c. STATIONARY STATES OF WELL-DEFINED ANGULAR MOMENTUM

Using the operators a_d and a_g, we can now go through the same arguments we used for a_x and a_y. It follows that the spectra of N_d and N_g are composed of all positive integers and zero. In addition, specifying a pair $\{ n_d, n_g \}$ of such integers determines uniquely (to within a constant factor) the eigenvector common to N_d and N_g, associated with these eigenvalues, which is written:

$$| \chi_{n_d,n_g} \rangle = \frac{1}{\sqrt{(n_d)!\,(n_g)!}} (a_d^\dagger)^{n_d} (a_g^\dagger)^{n_g} | \varphi_{0,0} \rangle \tag{47}$$

N_d and N_g therefore form a C.S.C.O. in $\mathscr{E}_{xy}$. Thus we see, by using (46), that $| \chi_{n_d,n_g} \rangle$ is also an eigenvector of H_{xy} and of L_z, with the eigenvalues $(n + 1)\hbar\omega$ and $m\hbar$, where n and m are given by:

$$\begin{aligned} n &= n_d + n_g \\ m &= n_d - n_g \end{aligned} \tag{48}$$

Equations (48) enable us to understand the origin of the name of right or left "circular quanta". The action of the operator $a_d^\dagger$ on $| \chi_{n_d,n_g} \rangle$ yields a state with one more quantum, to which, since m has increased by one, an additional angular momentum $+ \hbar$ must be attributed (this corresponds to a counterclockwise rotation about Oz). Similarly, $a_g^\dagger$ yields a state with one more quantum, of angular momentum $- \hbar$ (clockwise rotation).

Since n_d and n_g are positive integers (or zero), our results are in agreement with those of the preceding section : the eigenvalues of H_{xy} are of the form $(n + 1)\hbar\omega$, where n is a positive integer or zero; their degree of degeneracy is $(n + 1)$ since, for fixed n, we can have :

$$\begin{aligned} n_d &= n \quad\quad ; n_g = 0 \\ n_d &= n - 1 ; n_g = 1 \\ &\vdots \\ n_d &= 0 \quad\quad ; n_g = n \end{aligned} \tag{49}$$

Furthermore, we see that the eigenvalues of L_z are of the form $m\hbar$, where m is a positive or negative integer or zero, which is the result that was established for the general case in chapter VI. In addition, table (49) tells us which values of m are associated with a given value of n. For example, for the ground state, we have $n_d = n_g = 0$, and therefore, necessarily, $m = 0$; for the first excited state, we can have $n_d = 1$ and $n_g = 0$, or $n_d = 0$ and $n_g = 1$, which yields either $m = + 1$ or $m = - 1$. In general, formulas (48) and (49) show that, for a given energy level $(n + 1)\hbar\omega$, the possible values of m are:

$$m = n, n - 2, n - 4, ..., - n + 2, - n \tag{50}$$

It follows that, to a pair of values of n and m, there corresponds a single vector (to within a constant factor):

$$| \chi_{n_d = \frac{n+m}{2}, n_g = \frac{n-m}{2}} \rangle$$

H and L_z therefore form a C.S.C.O. in $\mathscr{E}_{xy}$.

COMMENT :

For a given value of the total energy (labeled by n), the states $| \chi_{n_d = n, n_g = 0} \rangle$ and $| \chi_{n_d = 0, n_g = n} \rangle$ therefore correspond to the maximal $(n\hbar)$ and minimal $(- n\hbar)$ values of L_z. These states therefore recall the classical right and left circular motions associated with a given value of the total energy, for which $\mathscr{L}_z$ takes on its maximal and minimal values (see §1-a).

d. WAVE FUNCTIONS ASSOCIATED WITH THE EIGENSTATES COMMON TO H_{xy} AND L_z

To conserve the symmetry of the problem with respect to rotation about Oz, we shall use polar coordinates, setting:

$$\begin{aligned} x &= \rho \cos \varphi \qquad \rho \geqslant 0 \\ y &= \rho \sin \varphi \qquad 0 \leqslant \varphi < 2\pi \end{aligned} \tag{51}$$

Now, what is the action of the operators a_d and a_g on a function of ρ and φ? We shall begin by determining their action on a function of x and y. Knowing that of X and P_x and therefore that of a_x (and, by analogy, that of a_y), we can use (40), which yields:

$$a_d \Longrightarrow \frac{1}{2}\left[\beta(x - iy) + \frac{1}{\beta}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\right] \tag{52}$$

According to the rules for differentiating functions of several variables, we then obtain:

$$a_d \Longrightarrow \frac{e^{-i\varphi}}{2}\left[\beta\rho + \frac{1}{\beta}\frac{\partial}{\partial\rho} - \frac{i}{\beta\rho}\frac{\partial}{\partial\varphi}\right] \tag{53}$$

Similarly:

$$a_d^\dagger \Longrightarrow \frac{e^{i\varphi}}{2}\left[\beta\rho - \frac{1}{\beta}\frac{\partial}{\partial\rho} - \frac{i}{\beta\rho}\frac{\partial}{\partial\varphi}\right] \tag{54}$$

and:

$$\begin{aligned} a_g &\Longrightarrow \frac{e^{i\varphi}}{2}\left[\beta\rho + \frac{1}{\beta}\frac{\partial}{\partial\rho} + \frac{i}{\beta\rho}\frac{\partial}{\partial\varphi}\right] \\ a_g^\dagger &\Longrightarrow \frac{e^{-i\varphi}}{2}\left[\beta\rho - \frac{1}{\beta}\frac{\partial}{\partial\rho} + \frac{i}{\beta\rho}\frac{\partial}{\partial\varphi}\right] \end{aligned} \tag{55}$$

To calculate the wave functions $\chi_{n_d,n_g}(\rho, \varphi)$, simply apply the differential operators which represent $a_d^\dagger$ and $a_g^\dagger$ to the function $\chi_{0,0}(\rho, \varphi)$, which is, according to (31):

$$\chi_{0,0}(\rho, \varphi) = \frac{\beta}{\sqrt{\pi}} e^{-\beta^2\rho^2/2} \tag{56}$$

Now it can be seen from (54) and (55) that the action of $a_d^\dagger$ (or of $a_g^\dagger$) on a function of the form $e^{im\varphi}F(\rho)$ is given by:

$$\begin{aligned} a_d^\dagger[e^{im\varphi}F(\rho)] &= \frac{e^{i(m+1)\varphi}}{2}\left[\left(\beta\rho + \frac{m}{\beta\rho}\right)F(\rho) - \frac{1}{\beta}\frac{dF}{d\rho}\right] \\ a_g^\dagger[e^{im\varphi}F(\rho)] &= \frac{e^{i(m-1)\varphi}}{2}\left[\left(\beta\rho - \frac{m}{\beta\rho}\right)F(\rho) - \frac{1}{\beta}\frac{dF}{d\rho}\right] \end{aligned} \tag{57}$$

Through the repeated application of these relations to (56), we see that the φ-dependence of $\chi_{n_d,n_g}(\rho, \varphi)$ is simply given by: $e^{i(n_d - n_g)\varphi}$. This is a general result, established in chapter VI: the φ-dependence of an eigenfunction of L_z of eigenvalue $m\hbar$ is $e^{im\varphi}$.

If, in (57), we choose $F(\rho) = \rho^m \, e^{-\beta^2\rho^2/2}$, then:

$$a_d^\dagger[e^{im\varphi}\rho^m \, e^{-\beta^2\rho^2/2}] = \beta e^{i(m+1)\varphi}\rho^{m+1} \, e^{-\beta^2\rho^2/2} \tag{58}$$

Applying the operator $a_d^\dagger$ to the function $\chi_{0,0}(\rho)$ n_d times, we obtain:

$$\chi_{n_d,0}(\rho, \varphi) = \frac{\beta}{\sqrt{\pi (n_d)!}} e^{in_d\varphi}(\beta\rho)^{n_d} \, e^{-\beta^2\rho^2/2} \tag{59}$$

An analogous calculation yields:

$$\chi_{0,n_g}(\rho, \varphi) = \frac{\beta}{\sqrt{\pi (n_g)!}} e^{-in_g\varphi}(\beta\rho)^{n_g} \, e^{-\beta^2\rho^2/2} \tag{60}$$

These wave functions are normalized. For a given energy level $(n + 1)\hbar\omega$, the wave functions (59) and (60) correspond to the limiting values $+ n$ and $- n$ of the quantum number m. Their ρ-dependence is particularly simple: their modulus reaches a maximum for $\rho = \sqrt{n}/\beta$. Therefore (as in the case of a one-dimensional harmonic oscillator), the spatial spread of these wave functions increases with the energy $(n + 1)\hbar\omega$ with which they are associated.

In the same way, application of the operators $a_d^\dagger$ (or $a_g^\dagger$) to (59) and (60) permits the construction of the functions $\chi_{n_d,n_g}(\rho, \varphi)$ for any n_d and n_g. The results obtained for the first excited levels are given in table I.

$$n = 0 \qquad m = 0 \qquad \chi_{0,0}(\rho) = \frac{\beta}{\sqrt{\pi}} e^{-\beta^2\rho^2/2}$$

$$n = 1 \begin{cases} m = 1 & \chi_{1,0}(\rho, \varphi) = \dfrac{\beta}{\sqrt{\pi}} \beta\rho \, e^{-\beta^2\rho^2/2} \, e^{i\varphi} \\ m = -1 & \chi_{0,1}(\rho, \varphi) = \dfrac{\beta}{\sqrt{\pi}} \beta\rho \, e^{-\beta^2\rho^2/2} \, e^{-i\varphi} \end{cases}$$

$$n = 2 \begin{cases} m = 2 & \chi_{2,0}(\rho, \varphi) = \dfrac{\beta}{\sqrt{2\pi}} (\beta\rho)^2 \, e^{-\beta^2\rho^2/2} \, e^{2i\varphi} \\ m = 0 & \chi_{1,1}(\rho, \varphi) = \dfrac{\beta}{\sqrt{\pi}} [(\beta\rho)^2 - 1] \, e^{-\beta^2\rho^2/2} \\ m = -2 & \chi_{0,2}(\rho, \varphi) = \dfrac{\beta}{\sqrt{2\pi}} (\beta\rho)^2 \, e^{-\beta^2\rho^2/2} \, e^{-2i\varphi} \end{cases}$$

TABLE I

Eigenfunctions common to the Hamiltonian H_{xy} and the observable L_z, for the first levels of the two-dimensional harmonic oscillator.

COMMENT:

The functions $\chi_{n_d,0}(\rho, \varphi)$ given in (59) are proportional to $e^{-\beta^2\rho^2/2}(\beta\rho\, e^{i\varphi})^{n_d}$. More generally, all their linear combinations are of the form:

$$F(\rho, \varphi) = e^{-\beta^2\rho^2/2} f(\beta\rho\, e^{i\varphi}) \tag{61}$$

(where f is an arbitrary function of one variable) and are eigenfunctions of N_g with the eigenvalue zero. It can easily be shown from (55) that:

$$a_g F(\rho, \varphi) = 0 \tag{62}$$

Similarly, the subspace of eigenfunctions of N_d of eigenvalue zero is composed of functions of the form:

$$G(\rho, \varphi) = e^{-\beta^2\rho^2/2} g(\beta\rho\, e^{-i\varphi}) \tag{63}$$

4. Quasi-classical states

Using the properties of the one-dimensional harmonic oscillator, we can easily calculate the time evolution of the state vector and the mean values of the various observables of the two-dimensional oscillator. For example, it is not difficult to show that in the mean values $\langle X \rangle(t)$ and $\langle Y \rangle(t)$, as well as $\langle P_x \rangle(t)$ and $\langle P_y \rangle(t)$, only the Bohr frequency ω appears. Moreover, it can be shown that these mean values exactly obey the classical equations of motion. In this section, we shall be concerned with the properties and evolution of the quasi-classical states of the two-dimensional harmonic oscillator.

a. DEFINITION OF THE STATES $| \alpha_x, \alpha_y \rangle$ AND $| \alpha_d, \alpha_g \rangle$

To construct a quasi-classical state of the two-dimensional harmonic oscillator, we can base our reasoning on the one-dimensional oscillator (*cf.* complement G_V). Recall that, in a quasi-classical state associated with a given classical motion, the mean values $\langle X \rangle(t)$ and $\langle P \rangle(t)$ coincide at each instant with $x(t)$ and $p(t)$. Similarly, the mean value of the Hamiltonian H is equal (to within a half-quantum $\hbar\omega/2$) to the classical energy. We showed in complement G_V that, at any time, the quasi-classical states are eigenstates of the destruction operator a and can be written:

$$| \alpha \rangle = \sum_n c_n(\alpha) \, | \varphi_n \rangle \tag{64}$$

where α is the eigenvalue of a, and:

$$c_n(\alpha) = \frac{\alpha^n}{\sqrt{n!}} e^{-|\alpha|^2/2} \tag{65}$$

In the case which concerns us here, we can use the rules of the tensor product to obtain the quasi-classical states in the form:

$$| \alpha_x, \alpha_y \rangle = | \alpha_x \rangle \otimes | \alpha_y \rangle = \sum_{n_x=0}^{\infty} \sum_{n_y=0}^{\infty} c_{n_x}(\alpha_x)\, c_{n_y}(\alpha_y) \, | \varphi_{n_x,n_y} \rangle \tag{66}$$