Hamiltonian systems

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial t}$$

$$= \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q_i} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}.$$

$$x = \begin{pmatrix} p \\ q \end{pmatrix}$$

$$\dot{x}_i = F_i(x,t) \quad \text{in general.}$$

but

$$\dot{x}_i = S_{ij}\frac{\partial H}{\partial x_j} \qquad S = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

very special

$$\nabla \cdot F = \partial_i F_i = \frac{\partial}{\partial x_i} S_{ij} \frac{\partial H}{\partial x_j} = 0$$

Consider $\Omega = \int_V d^{2N}x$ as points in V move with time

$$\dot{\Omega} = \frac{d}{dt}\int_V d^{2N}x = \oint \dot{x}_i \, dS_i = \oint F_i \, dS_i = \int \nabla \cdot F \, d^{2N}x = 0.$$

Liouville $\Rightarrow$ no attractors

Symplectic property

$$\frac{d}{dt}\left(\delta p_i' \delta q_i - \delta q_i' \delta p_i\right) = 0$$

$$= \delta \dot{p}_i \delta q_i' + \delta p_i \delta \dot{q}_i' - \delta \dot{q}_i \delta p_i' - \delta q_i \delta \dot{p}_i'$$

$$= \frac{d}{dt}\left(\delta x_i \, S_{ij} \, \delta x_j'\right) = \frac{d}{dt} (\delta p \, \delta q) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \delta p' \\ \delta q' \end{pmatrix}$$

$$= \delta \dot{x}_i \, S_{ij} \, \delta x_j' + \delta x_i \, S_{ij} \, \delta \dot{x}_j'$$

Now

$$\delta \dot{x}_i = \delta F_i = \frac{\partial F_i}{\partial x_k} \delta x_k$$

$$= \frac{\partial}{\partial x_k}\left(S_{il} \frac{\partial H}{\partial x_l}\right) \delta x_k = S_{il} \frac{\partial^2 H}{\partial x_k \partial x_l} \delta x_k$$

So

$$\frac{d}{dt}\left(\delta x_i S_{ij} \delta x_j'\right) = S_{il} \frac{\partial^2 H}{\partial x_k \partial x_l} \delta x_k \delta x_j' \, S_{ij}$$

$$+ \delta x_i \, S_{ij} \, S_{jn} \frac{\partial^2 H}{\partial x_n \partial x_v} \delta x_v'$$

Now

$$S_{ij} S_{jn} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} = \begin{pmatrix} -I & \\ & -I \end{pmatrix} = -\delta_{in}$$

$$S_{il} S_{ij} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = \delta_{lj}$$

So

$$\frac{d}{dt}\left(\delta x_i S_{ij} \delta x_j'\right)$$

$$= -\delta_{\ell j} \frac{\partial^2 H}{\partial x_k \partial x_\ell} \delta x_k \delta x_j'$$

$$- \delta_{iu} \delta x_i \frac{\partial^2 H}{\partial x_u \partial x_v} \delta x_v'$$

$$= \frac{\partial^2 H}{\partial x_k \partial x_j} \delta x_k \delta x_j'$$

$$- \frac{\partial^2 H}{\partial x_i \partial x_v} \delta x_i \delta x_v' = 0$$

Now

$$\delta p_i' \delta q_i - \delta q_i' \delta p_i = \delta x_i S_{ij} \delta x_j'$$

$= \pm$ area of $\begin{pmatrix}\delta p_i' \\ \delta q_i'\end{pmatrix}$, $\begin{pmatrix}\delta p_i \\ \delta q_i\end{pmatrix}$

It's like 3 component of

$$\left[(x,y,z) \otimes (x',y',z')\right]_3 = xy' - yx',$$

$\delta x_i S_{ij} \delta x'_j$ is diff form of Poincaré's integral invariant

$$\oint_{\gamma(t)} p_i \, dq_i$$

which is independent of t if $\gamma(t)$ follows flow.

$$\frac{d}{dt} \oint_{\gamma(t)} p_i \, dq_i = \oint \dot{p}_i \, dq_i + \oint p_i \, d\dot{q}_i$$

$$= \oint \dot{p}_i \, dq_i - \oint \dot{q}_i \, dp_i + \underbrace{\oint d(p_i \dot{q}_i)}_{=0}$$

$$= \oint -\frac{\partial H}{\partial q_i} dq_i - \oint \frac{\partial H}{\partial p_i} dp_i + 0$$

$$= \oint dH = 0.$$

all at fixed time t.

$$\oint p(s,t(s)) \cdot dq(s,t(s)) - H(p,q,t(s))\, dt(s)$$

Canonical T's.

$$\bar{p} = g_1(p,q,t)$$

$$\bar{q} = g_2(p\,q,t)$$

$$\dot{\bar{p}} = -\frac{\partial \bar{H}}{\partial \bar{q}}$$

$$\dot{\bar{q}} = \frac{\partial \bar{H}}{\partial \bar{p}}.$$

$S(\bar{p}, q, t)$ generating fn.

$$\bar{q} = \frac{\partial S(\bar{p}\,q\,t)}{\partial \bar{p}} \qquad p = \frac{\partial S(\bar{p}\,q\,t)}{\partial q}$$

One may show it's symplectic.

$$\delta p \cdot \delta q' - \delta q \cdot \delta p' = \delta \bar{p} \cdot \delta \bar{q}' - \delta \bar{q} \cdot \delta \bar{p}'$$

$$\bar{H} = H + \frac{\partial S}{\partial t}$$

hamiltonian maps.

$$M_i(x(t),t) = x_i(t+T)$$

$$\frac{\partial M_i}{\partial x_j}\,\delta x_j = \delta x_i(t+T)$$

symplectic condition:

$$\delta x_i(t+T)\, S_{ij}\, \delta x_j'(t+T) = \delta_i(t)\, S_{ij}\, \delta x_j'(t)$$

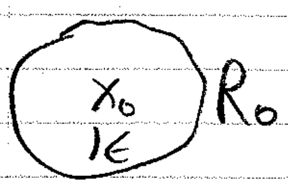
$$\frac{\partial M_i}{\partial x_k}\,\delta x_k\, S_{ij}\, \frac{\partial M_j}{\partial x_l}\,\delta x_l$$

$$(M_{i,k})^T\, S\, M = S$$

So $\frac{\partial M_i}{\partial x_k}$ is a symplectic matrix.

Poincaré recurrence thm.

$H = H(p,q)$ no t



integrable system. $H = H(p,q)$ no t

$$\dot f = \frac{\partial f}{\partial p_i}\dot p_i + \frac{\partial f}{\partial q_i}\dot q_i + \cancel{\frac{\partial f}{\partial t}} \qquad f(p,q) \text{ no } t$$

$$= \frac{\partial f}{\partial p_i}\left(-\frac{\partial H}{\partial q_i}\right) + \frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} = [f,H]$$

So

$$[f,H] = 0 \iff \dot f = 0$$

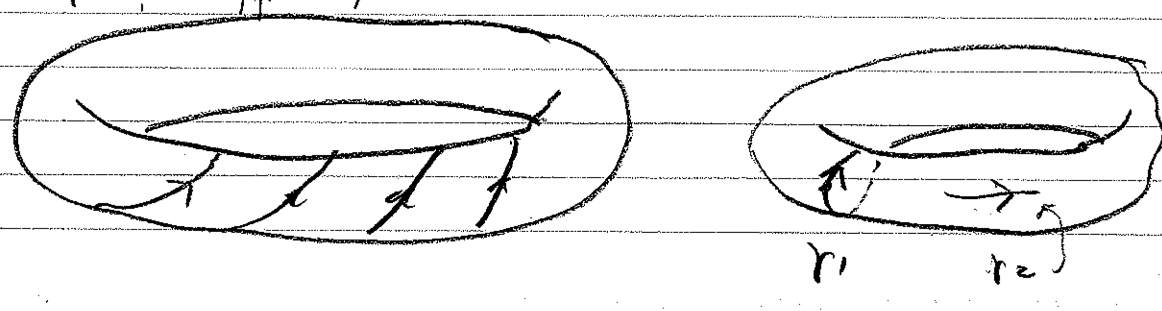
$$[H,H] = 0.$$

A time-independent H sys. is <u>integrable</u> if it has N <u>independent</u> global constants of motion $f_i(p,q)$ and if

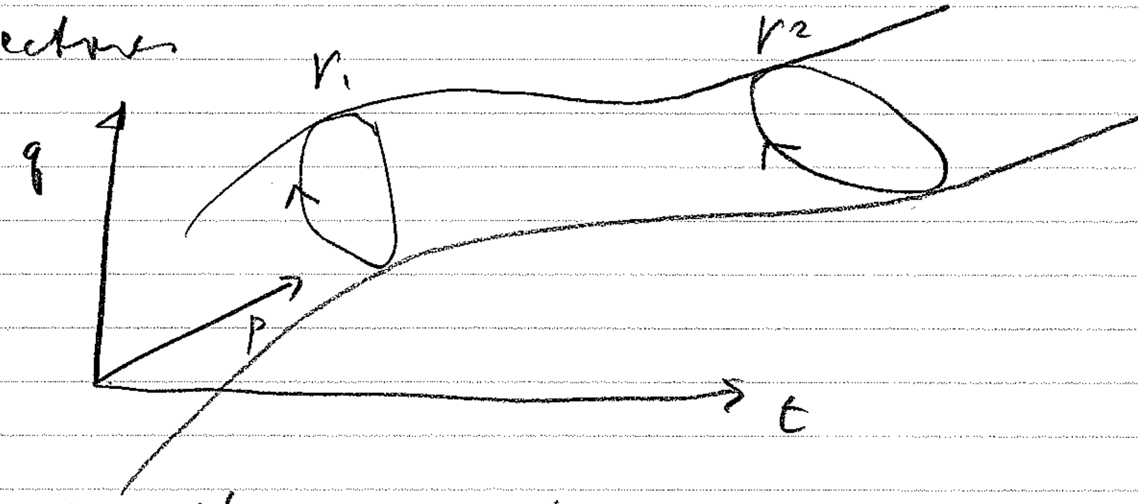
$$[f_i, f_j] = 0.$$

The f_i are <u>in involution</u>.

One may show that motion $f_i(p,q) = k_i$ is on an N-dim torus T^N. If $N=2$



Suppose two curves γ_1 and γ_2 encircle the same tube of phase-space trajectories



of a hamiltonian system

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad \dot{q}_i = \frac{\partial H}{\partial p_i} .$$

Then the integrals of the form $p\,dq - H\,dt$ along them are the same:

$$\oint_{\gamma_1} p_i\,dq_i - H\,dt = \oint_{\gamma_2} p_i\,dq_i - H\,dt .$$

(See Sec. 44 of Arnold's Math Meth of Class Mech)

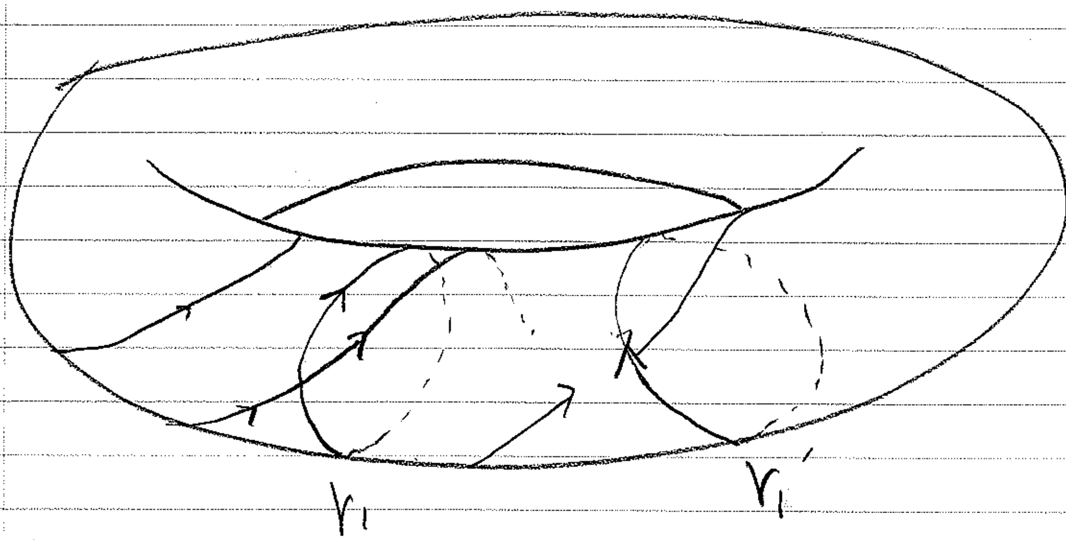
Suppose now that $\frac{\partial H}{\partial t} = 0$, i.e. that $H = H(p,q)$. Then if γ_1 and γ_2 lie on the same surface of constant H

$$\oint_{\gamma_1} p_i\,dq = \oint_{\gamma_2} p_i\,dq \text{ since } \oint H\,dt = H\oint dt = 0$$

time-independent

Note that for an integrable system, each trajectory lie on the N-dimensional surface of an N-torus T^N on which

$f_i(p,q) = h_i$ are fixed.

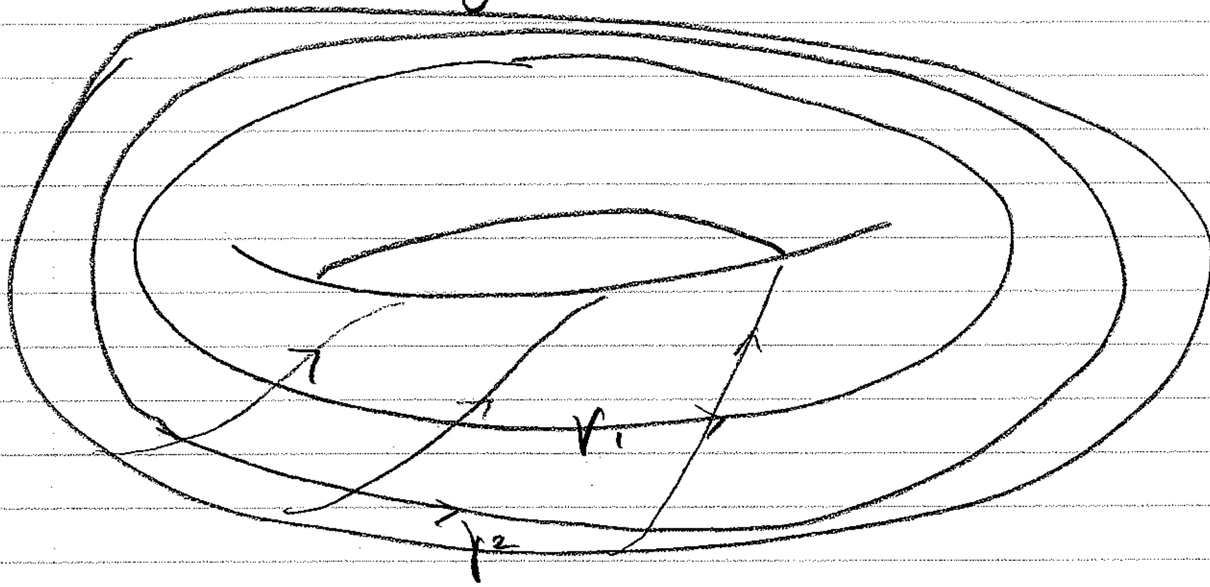


the integrals around γ_1 and γ_1' of

$$\oint_{\gamma_1} p_i dq_i = \oint_{\gamma_1'} p_i dq_i$$

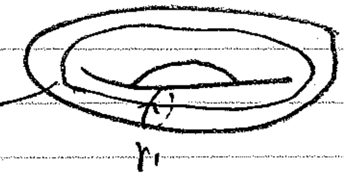
even for arbitrary deformations in p, q, and t as long as both stay on the same surface of constant H.

Similarly,



$$\oint_{\gamma_2} p_i \, dq_i = \oint_{\gamma_2'} p_i \, dq_i$$

An N-torus T^N is a surface whose points can be labeled by N angles. In the 2-torus



the angles θ_1 and θ_2 tell how far one is along γ_1 and γ_2.

If $n\omega_1 + m\omega_2 = 0$, then $\quad n\omega_1 = -m\omega_2 \quad \frac{n}{m} = -\frac{\omega_2}{\omega_1}$

$$\omega_1 = -\frac{m}{n}\omega_2 = -m\,\omega_0$$

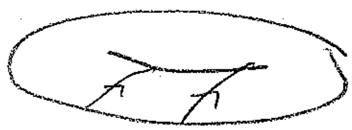
$$\omega_2 = n\,\omega_0$$

$\Updownarrow$ $\frac{\omega_2}{\omega_1}$ rational

So

$$\vec{\omega} = \vec{m}\,\omega_0.$$

the ~~orbits~~ is periodic



and does not cover torus.

But if there are no two integers n, m s.t.

$$n\omega_1 + m\omega_2 = 0$$

then the ratio $-\omega_2/\omega_1$ is not rational.

The motion is quasi periodic and fills torus.

This generalizes to $2N$ dimensions & the N-torus T^N. Now typically, the ~~orbits~~ orbits are quasiperiodic & fill the tori. Occasionally, they are periodic.

Periodic orbits have zero measure but are dense in phase space.

So an N-torus T^N has N angles $\theta_1 \ldots \theta_N$ and N irreducible paths γ_i.

The orbits of a $(\partial H/\partial t = 0)$ integrable system of $2N$ variables lie on N-tori and have N constants of the motion

$$f_i(pq) = h_i .$$

We can make a canonical transformation to new coordinates $\bar{p}\,\bar{q}$ such that $\bar{H} = \bar{H}(\bar{p},\bar{q}) = \bar{H}(\bar{p})$. One may choose the $\bar{p}$ as the f's

$$\bar{p}_i = f_i(p,q)$$

So

$$0 = \dot{\bar{p}}_i = \frac{\partial \bar{H}}{\partial \bar{q}_i} = 0.$$

So $\bar{H} = \bar{H}(\bar{p})$. Many ways to do this.

Action-angle variables are convenient.

$(\bar{p},\bar{q}) = (I,\theta)$ where

$$\bar{p}_i = I_i = \frac{1}{2\pi}\oint_{\gamma_i} p_j \cdot dq_j$$ around each of N irred. paths of T^N.

Note that

$$I_i = \oint_{\gamma_i} \frac{p_j\, dq_j}{2\pi} = \oint_{\gamma_i'} \frac{p_j \cdot dq_j}{2\pi}$$

in view of P-C thm. Now our C.T. is

$$\bar{q}_i = \frac{\partial S(\bar{p}, q, t)}{\partial \bar{p}_i} \qquad p_i = \frac{\partial (\bar{p}, q, t)}{\partial q_i}$$

So here on one circuit γ_i

$$\oint_{\gamma_i} p_j\, dq_j = \oint_{\gamma_i} \frac{\partial S(I, q)}{\partial q_i} dq_i = \Delta_i S$$

$$\|$$

$$2\pi I_i = \Delta_i S$$

And also

$$\bar{q}_i = \theta_i = \frac{\partial S(I, q)}{\partial \bar{p}_i} = \frac{\partial S(I, q)}{\partial I_i}$$

So on one circuit γ_i

$$\Delta_i \theta_j = \frac{\partial \Delta_i S}{\partial I_j} = \frac{\partial\, 2\pi I_i}{\partial I_j} = 2\pi \delta_{ij}.$$

$$\dot{I}_i = \frac{\partial \bar{H}}{\partial \theta_i} = 0 \qquad \dot{\theta}_i = \frac{\partial \bar{H}}{\partial I_i} = \omega_i(I).$$

So $\qquad \theta_i(t) = \theta_i(0) + \omega_i(I)\, t.$

Can Trans

$$\bar q = \frac{\partial S(\bar p q t)}{\partial \bar p} \qquad p = \frac{\partial S(\bar p q t)}{\partial q}$$

$$\bar H(\bar p \bar q t) = H(p q t) + \frac{\partial S}{\partial t}$$

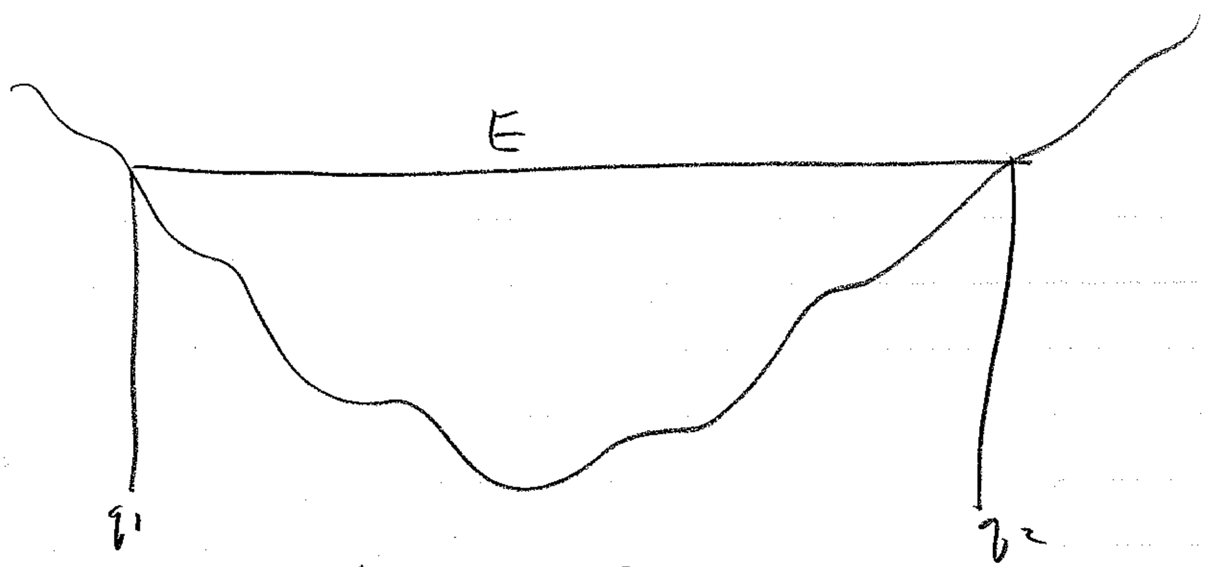
So

$$\bar H(I) = H\left(\frac{\partial S}{\partial q}, q\right)$$

which is the H-J eq'n.

Ex

$$H = \frac{p^2}{2m} + V(q)$$



$$I = \frac{1}{\pi}\int_{q_1}^{q_2} p\,dq = \frac{1}{\pi}\int_{q_1}^{q_2}\sqrt{2m(E - V(q))}\,dq$$

If $\quad V(q) = \frac{1}{2} m \omega^2 q^2$,

then $\quad q_2 = -q_1$ and $\quad \frac{1}{2} m\omega^2 q_1^2 = E$

$$q_2 = \sqrt{\frac{2E}{m\omega^2}}$$

and doing integral, we get

$$I = \frac{E}{\omega}$$

So $\quad \bar{H}(I) = \omega I$

$$\omega(I) = \frac{d\bar{H}}{dI} = \omega \quad \text{indep of } I$$

$$\theta(t) = \theta(0) + \omega t.$$

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2} m\omega^2 q^2 = \omega I$$

$$\left(\frac{\partial S}{\partial q}\right)^2 = \left[\omega I - \frac{1}{2} m\omega^2 q^2\right] 2m$$

$$\frac{\partial S}{\partial q} = \sqrt{2m\left(\omega I - \frac{1}{2} m\omega^2 q^2\right)} = p$$

$$S = \int \sqrt{2m\left(\omega I - \frac{1}{2} m\omega^2 q^2\right)}\, dq$$

doing integral and using

$$\theta = \frac{\partial S}{\partial I} \qquad p = \frac{\partial S}{\partial q}$$

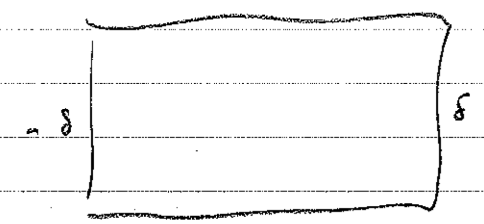
we get

$$q = \left(\frac{2I}{m\omega}\right)^{1/2} \cos\theta$$

$$p = -(2mI\omega)^{1/2} \sin\theta .$$

ellipse period $2\pi/\omega$ a 1-Torus T^1

For $N>1$ N frequencies typically quasiperiodic.



Now $I = \frac{1}{2\pi}\oint p\,dq = \frac{4}{2\pi}(2mE\delta)^{1/2}$

$$\tilde{H}(I) = \frac{(\pi I)^2}{8m\delta^2}$$

$$\omega(I) = \frac{\pi^2 I}{4m\delta^2}$$

KAM:

The $w_i(\vec{I})$ in general are quasi-periodic. If ~~a~~ a vector $\vec{m}$ of integers gives

$$0 = \sum m_i w_i(I).$$

These are resonant tori.

They are destroyed by perturbation.

They are dense in phase space of compact. M.

But there is a very large set of "very nonresonant tori"

$$|m \cdot \omega| > K(\omega)\, |m|^{-(N+1)}$$

$$\left(\sum |m_i|\right)$$

for all $\vec{m}$. The set of tori that fails this condition has $L(0) = 0$