CHAPTER 4

PROFIT MAXIMIZATION

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4.1 UNCONSTRAINED MAXIMA AND MINIMA: FIRST-ORDER NECESSARY CONDITIONS

Postulates of purposeful behavior lead naturally to the specification of mathematical models that involve the maximization of some function of several variables. Most often, this maximization takes place subject to test conditions specifying constraints on the movements of the variables in addition to the specifications of values of parameters. The well-known model of utility maximization is an example of such a model: The consumer is asserted to maximize a utility function subject to the condition that he or she not exceed a given budgetary expenditure. There are some important examples, however, of unconstrained maximization, such as the model of a profit-maximizing firm (which will be dealt with below). Since the unconstrained case is simpler, we begin the analysis there.

In models with just one independent variable, the first-order condition necessary for \( y = f(x) \) to attain a stationary value is \( dy/dx = f'(x) = 0 \). That is, the line tangent to the curve \( f(x) \) must be horizontal at the stationary point. The term stationary point rather than maximum or minimum is appropriate at this juncture. The property of having a horizontal tangent line is common to the functions \( y = x^2 \), \( y = -x^2 \), and \( y = x^3 \) at the point \( x = 0 \), \( y = 0 \). The first function has a minimum at the origin, the second, a maximum, and the third, neither. However, it is clear that if the slope of the tangent line is not 0 (horizontal), then the function certainly cannot have either a maximum or a minimum. Hence \( f'(x) = 0 \) is a necessary but not sufficient condition for \( y = f(x) \) to have a maximum (or minimum) value.

Suppose now that \( y \) is a function of two variables, that is, \( y = f(x_1, x_2) \). What are the analogous necessary conditions for a maximum of this function? Proceeding...
intuitively from the case of one variable, it must necessarily be the case that at the point in question, the tangent plane must be horizontal. In order for the tangent plane to be horizontal, the first partials $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}$ must be 0; that is, the function must be level in the $x_1$ and $x_2$ directions.

Because intuition, especially about the second-order conditions for maximization, is often unreliable, the preceding argument will now be developed more rigorously. Let $y = f(x_1, x_2)$, and suppose we wish to consider the behavior of this function at some point $x^0 = (x_1^0, x_2^0)$. Instead of working with the whole function, however, consider the function evaluated along any (differentiable) curve that passes through the point $x^0$. The reason for doing this is that it will enable us to convert a problem in two variables to one involving one variable only, a problem we already know how to solve. All such curves can be represented parametrically by $x_1 = x_1(t)$, $x_2 = x_2(t)$, with $x_1 = x_1^0$, $x_2 = x_2^0$ at $t = 0$. That is, as $t$ varies in value, $x_1$ and $x_2$ vary, and hence the pair $[x_1(t), x_2(t)]$, denoted $x(t)$, traces out the locus of some curve in the $x_1x_2$ plane. [Setting $x_1(0) = x_1^0$, $x_2(0) = x_2^0$ merely ensures that the curve passes through $(x_1^0, x_2^0)$ for some value of $t$.]

**Example 1.** This parametric representation of a curve in the $x_1x_2$ plane was developed in Chap. 3. Again, suppose

$$\begin{align*} x_1 &= x_1^0 + h_1 t \\ x_2 &= x_2^0 + h_2 t \end{align*}$$

where $h_1$ and $h_2$ are arbitrary constants. Then these equations represent the straight lines in the $x_1x_2$ plane which pass through $(x_1^0, x_2^0)$. Any such line can be generated by appropriate choice of $h_1$ and $h_2$.

**Example 2.** Let

$$\begin{align*} x_1 &= x_1^0 + t \\ x_2 &= x_2^0 e^t \end{align*}$$

This parameterization represents an exponential curve. When $t = 0$, $x_1 = x_1^0$, $x_2 = x_2^0$; hence the curve passes through $(x_1^0, x_2^0)$.

**Example 3.** A parameterization that occurs frequently in the physical sciences is

$$\begin{align*} x &= a \cos \theta \\ y &= a \sin \theta \end{align*}$$

where $0 \leq \theta \leq 2\pi$. This represents the equation of a circle in the $xy$ plane, with radius $a$ and center at the origin.

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1 We will often find it convenient to use the vector notation $\mathbf{x} = (x_1, \ldots, x_n)$, where the single symbol $\mathbf{x}$ denotes multidimensional value.
The function \( f(x_1, x_2) \) evaluated along some differentiable curve \( x(t) = (x_1(t), x_2(t)) \) is \( y(t) = f(x_1(t), x_2(t)) \). If \( f(x_1, x_2) \) is to achieve a maximum value at \( x = x^0 \), the function evaluated along all such curves must necessarily have a maximum. Hence \( y(t) \) must have a maximum (at \( t = 0 \)) for all curves \( x(t) \). But the condition for this is simply \( y'(t) = 0 \). Using this chain rule the first-order conditions for a maximum are therefore

\[
\frac{dy}{dt} = y'(t) = f_1 \frac{dx_1}{dt} + f_2 \frac{dx_2}{dt} = 0 \tag{4-1}
\]

However, \( dy/dt \) must be 0 for all curves \( (x_1(t), x_2(t)) \) passing through \( x^0 \); i.e., for all values of \( dx_1/dt \) and \( dx_2/dt \). That is, it must be possible to put any values of \( dx_1/dt, dx_2/dt \) into this relationship and still obtain \( dy/dt = 0 \). The only way this can be guaranteed is if \( f_1 = f_2 = 0 \). Hence a necessary condition for \( f(x_1, x_2) \) to be maximized at \( x^0_1, x^0_2 \) is that the first partials of that function must be 0 at this point. The preceding conditions are, of course, only necessary conditions for \( y \) to achieve a stationary point; only the second derivative of \( y(t) \) reveals whether \( (x^0_1, x^0_2) \) is in fact a maximum, a minimum, or neither.

The generalization to the \( n \) variable case is direct, and the derivation is identical to the preceding. For \( y = f(x_1, x_2, \ldots, x_n) \) to be maximized at \( x^0 = (x^0_1, \ldots, x^0_n) \) it is necessary that all the first partial derivatives equal 0; that is, \( f_i = 0, i = 1, \ldots, n \).

### 4.2 SUFFICIENT CONDITIONS FOR MAXIMA AND MINIMA: TWO VARIABLES

For functions of one variable, \( y = f(x) \), a sufficient condition for \( f(x) \) to have a maximum at \( x = x^0 \) is that, together with \( f'(x^0) = 0, f''(x^0) < 0 \). The condition \( f''(x^0) < 0 \) expresses the notion that the slope is decreasing, e.g., as one walked over the top of a hill, the ground would be first rising, then level at the top, then falling. Alternatively, the function is called concave downward, or simply, concave, if \( f''(x) \leq 0 \). If \( f(x_1, x_2) \) has a maximum at \( x^0 \), then \( y(t) = f(x_1(t), x_2(t)) \) has a maximum for all curves \( x(t) \). Hence it must be the case that at the maximum point, 

\[
d^2y/dt^2 = y''(t) \leq 0 \text{ for all such curves.}
\]

The issues here are considerably more subtle than one might perceive at this point, as the next section will demonstrate. Although \( y''(t) \leq 0 \) is necessary for a maximum, it is not sufficient. By expanding \( f(x_1, x_2) \) by a Taylor series for functions of two (or, more generally, \( n \) variables), it can be shown that if \( y''(t) < 0 \) at \( t = 0 \) (the maximum point), then the function \( f(x_1, x_2) \) is strictly concave at \( (x^0_1, x^0_2) \). Thus, in that case, a maximum will be achieved if \( f_1 = f_2 = 0 \). This analysis will be presented in the appendix to this chapter.

Let us then evaluate \( y''(t) \). Using the chain and product rules on Eq. (4-1),

\[
y'(t) = f_1 x'_1(t) + f_2 x'_2(t)
\]
ifiable curve \( x(t) \) may have a maximum value if necessarily have a 1 curves \( x(t) \). But the first-order conditions

\[
(4-1)
\]

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one obtains (this was derived explicitly in Chap. 3)

\[
\frac{d^2 y}{dt^2} = f_1 \frac{d^2 x_1}{dt^2} + f_2 \frac{d^2 x_2}{dt^2} + f_{11} \left( \frac{dx_1}{dt} \right)^2 + 2 f_{12} \frac{dx_1}{dt} \frac{dx_2}{dt} + f_{22} \left( \frac{dx_2}{dt} \right)^2
\]

However, this is evaluated at \( (x_1, x_2) = (x_1^0, x_2^0) \), a stationary point; hence \( f_1 = f_2 = 0 \).

Letting \( h_1 = \frac{dx_1}{dt}, \ h_2 = \frac{dx_2}{dt} \) for notational convenience, the condition that \( \frac{d^2 y}{dt^2} < 0 \) for all curves passing through \( (x_1^0, x_2^0) \) means that

\[
f_{11} h_1^2 + 2 f_{12} h_1 h_2 + f_{22} h_2^2 < 0 \quad (4-2)
\]

for all values of \( h_1 \) and \( h_2 \) (except \( h_1 = h_2 = 0 \)). This inequality, since it must hold for all nontrivial \( h_1, h_2 \) (i.e., not both equal to 0), imposes restrictions on the signs and relative magnitudes of the second-order partials.

It is apparent from expression (4-2) that both \( f_{11} \) and \( f_{22} \) must be negative: Let \( h_2 = 0 \) and \( h_1 \) be any number and suppose \( f_{11} \) is positive. Then \( \frac{d^2 y}{dt^2} = f_{11} h_1^2 > 0 \), violating the sufficient conditions for a maximum. Interchanging all the subscripts gives the desired restriction on \( f_{22} \), as the formulation is completely symmetrical. Thus, in order to have \( \frac{d^2 y}{dt^2} < 0 \) at \( x^0 = (x_1^0, x_2^0) \), it is necessary that

\[
f_{11}(x^0) < 0 \quad \text{and} \quad f_{22}(x^0) < 0
\]

However, these conditions, which one might have guessed at by considering the one-variable case, are not, by themselves, sufficient for \( f(x_1, x_2) \) to have a maximum. We have yet to consider the role of the cross-partial \( f_{12} \) in this analysis. An additional restriction on the \( f_{ij} \)'s is required to ensure \( \frac{d^2 y}{dt^2} < 0 \) for all nontrivial \( h_1 \) and \( h_2 \). It can be derived by using the technique known as completing the square.

Consider the expression \( x^2 + 2bx \). If the term \( b^2 \) is both added and subtracted, the identity \( x^2 + 2bx = (x + b)^2 - b^2 \) results. Take Eq. (4-2) and factor out \( f_{11} \):

\[
f_{11} \left( h_1^2 + \frac{2 f_{12} h_2}{f_{11}} h_1 + f_{22} h_2^2 \right) < 0
\]

The first two terms in parentheses are quadratic in \( h_1 \) in the same sense as the preceding algebraic example. Completing the square in \( h_1 \) is accomplished by adding and subtracting \( \left( \frac{f_{12} h_2}{f_{11}} \right)^2 \) in the parentheses. This yields

\[
f_{11} \left[ \left( h_1 + \frac{f_{12} h_2}{f_{11}} \right)^2 + \left( h_2 \right)^2 \left( f_{11} f_{22} - f_{12}^2 \right) \right] < 0
\]

Since \( f_{11} < 0 \), in order to guarantee \( \frac{d^2 y}{dt^2} < 0 \), the square-bracketed term must be positive. However, the first term in the bracket is a squared term and hence is always positive anyway. In order to guarantee that \( \frac{d^2 y}{dt^2} < 0 \) for all values of \( h_1 \) and \( h_2 \), we must also require that the second term, in particular \( f_{11} f_{22} - f_{12}^2 \), be positive.
To sum up, then, suppose \( f(x_1, x_2) \) has a stationary point \( x = x^0 \), that is, the first-order necessary conditions for an extremum occur:

\[
f_1(x^0) = f_2(x^0) = 0
\]  

(4-3)

If, in addition,

\[
f_{11} < 0 \quad \text{and} \quad f_{11}f_{22} - f_{12}^2 > 0 \quad \text{evaluated at } x^0
\]  

(4-4)

a maximum position is assured. Note that if (4-4) is satisfied, \( f_{12} < 0 \) is implied. It is also important to note that condition (4-4) imposes a restriction only on the relative magnitude of \( f_{12} \); it does not imply anything about the sign of this second partial. The sign of \( f_{12} \) is thus irrelevant in determining whether a function has a maximum or minimum.

For \( f(x_1(t), x_2(t)) \) to achieve a minimum at \( x^0 = (x_1^0, x_2^0) \) the same first-order conditions (4-3) must, of course, be met. The analogous sufficient second-order conditions, i.e., guaranteeing \( d^2y/dt^2 > 0 \), are

\[
f_{11} > 0, \quad f_{22} > 0 \quad \text{and} \quad f_{11}f_{22} - f_{12}^2 > 0
\]  

(4-5)

where all partials are evaluated at \( x^0 \). Note that the term \( f_{11}f_{22} - f_{12}^2 \) is positive for both minima and maxima. If this term is found to be negative, then the surface has a “saddle” shape at \( x^0 \); it rises in one direction and falls in another, similar to the point in the center of a saddle.

One last precautionary note must be mentioned. These second-order conditions are sufficient conditions for a maximum or minimum; the strict inequalities (4-4) and (4-5) are not implied by maxima and minima. For example, the function \( y = x^4 \) has a maximum at the origin, yet its second derivative is 0 there. Likewise \( y = x^3 \) has neither a maximum nor a minimum at \( x = 0 \), yet its second derivative is also 0 there. Hence, if one or more of the relations in (4-4) or (4-5) hold as equalities, the observer is unable at that juncture to determine the shape of the function at that point. The general rule, which will not be proved here, is if \( d^2y/dt^2 = 0 \) for some \( x(t) \), one must calculate the higher-order derivatives \( d^3y/dt^3 \), \( d^4y/dt^4 \), et cetera. Then if the first occurrence of \( d^n y/dt^n < 0 \) for all curves \( x(t) \) is an even order \( n \), then the function has a maximum (minimum, if \( > 0 \)), whereas if that first occurrence happens for an odd number \( n \), neither a maximum nor a minimum is achieved. To make matters worse, however, there are functions, for example, \( y = e^{-1/x^2} \), which have a minimum, say, at some point (here, \( x = 0 \)), and yet the derivatives of all finite orders are 0 at that point (for this function, at \( x = 0 \)). We shall ignore all such “nonregular” situations in which the ordinary sufficient conditions for an extremum do not hold; we will confine our attention only to “regular” extrema.

It can be shown that the second-order conditions (4-4) are sufficient for a function to be concave (downward) at points other than a stationary value. Likewise, (4-5) guarantees that the function is convex (i.e., concave upward) at any point. Proof of these propositions will be deferred to the appendix.
Example 1. Suppose \( f(x_1, x_2) \) has a maximum at some point. Then the sufficient second-order conditions are, again,
\[
f_{11}h_1^2 + 2f_{12}h_1h_2 + f_{22}h_2^2 < 0
\]
for all nontrivial values of \( h_1 \) and \( h_2 \). Since this holds for all values of \( h_1 \) and \( h_2 \), suppose we let \( h_1 = 1, h_2 = \pm 1 \). Then this condition implies
\[
f_{11} + f_{22} \pm 2f_{12} < 0
\]
or
\[
f_{11} + f_{22} < \mp 2f_{12}
\]
Since \( f_{11} \) and \( f_{22} \) are both negative,
\[
|f_{11} + f_{22}| > 2|f_{12}|
\]
is implied by the sufficient second-order conditions for a maximum.

Example 2. Suppose \( f(x_1, x_2) \) is strictly concave at some point. The sufficient condition for concavity is again Eq. (4-2),
\[
f_{11}h_1^2 + 2f_{12}h_1h_2 + f_{22}h_2^2 < 0
\]
Now let \( h_1 = f_{11}, h_2 = -f_1 \). Then (4-2) implies
\[
f_1f_3^2 - 2f_{12}f_1f_2 + f_{22}f_1^2 < 0
\]
This was the condition developed in Chap. 3 [Eq. (3-24)] for the level curves to be convex to the origin. Hence concavity implies level curves having this property. The converse, however, is false.

Example 3 (Monopolistic price discrimination). In a practice called price discrimination, monopolists are sometimes able to charge different consumers different prices for the same service. When government regulation gave certain airlines near-monopoly rights over certain routes, the airline industry discovered that some of its customers were businesspersons, eager to make some meeting in a distant city for a day or two, while other of its customers were more likely planning vacations far in advance of their trips. The business travelers' demands were likely less elastic than those of the families, and the airlines sought to exploit that difference. The airlines found that self-selection would occur by giving discounts for tickets that required a Saturday stayover, something businesspersons rarely did. (These practices persist in the absence of route regulation, apparently due to the scarcity of gates at airports, which effectively restricts entry in many markets.)

A discriminating monopolist faces two market demands, \( p_1 = p_1(x_1) \) and \( p_2 = p_2(x_2) \). Its objective is to maximize
\[
\pi = R_1(x_1) + R_2(x_2) - C(x_1 + x_2)
\]
where \( R_1(x_1) \) and \( R_2(x_2) \) are the total revenue functions \( p_1(x_1)x_1 \) and \( p_2(x_2)x_2 \), respectively. The derivatives of the total revenue functions are the marginal revenues \( MR_1(x_1) \) and \( MR_2(x_2) \), respectively. Note also that total cost \( C \) is really a function
of one variable, \( x = x_1 + x_2 \), and that \( C_1 = C_2 = C'(x) = MC(x) \). The necessary first-order conditions are

\[
\begin{align*}
\pi_1 &= MR_1(x_1) - MC(x) = 0 \\
\pi_2 &= MR_2(x_2) - MC(x) = 0
\end{align*}
\] (4-8a)

These conditions imply that maximum profits occur when the firm sets the marginal revenues in each market equal to each other and equal to marginal cost:

\[MR_1(x_1) = MR_2(x_2) = MC(x)\]

The intuition should be clear. If the marginal revenue is $100 in market 1 and only $50 in market 2, the firm would increase profits by shifting sales from market 2 to market 1. The common value of marginal revenue must then equal marginal cost; if \( MR > MC \), the firm could increase profits by increasing output, and so forth.

The sufficient second-order conditions are

\[
\begin{align*}
\pi_{11} &= MR_1'(x_1) - MC'(x) < 0 \\
\pi_{22} &= MR_2'(x_2) - MC'(x) < 0
\end{align*}
\] (4-9a)

and

\[\pi_{11} \pi_{22} - \pi_{12}^2 > 0 \] (4-9c)

Relations (4-9a) and (4-9b) state that the \( MC \) curve must cut each \( MR \) curve from below, else, in each market, the firm would increase profits by increasing output beyond where \( MR = MC \). We leave it as an exercise to show that (4-9c) means that the \( MC \) curve must cut the lateral sum of the \( MR \) curves from below.

Consider now the first-order conditions (4-8). Marginal revenue is related to the elasticity of the demand curve by the formula \( MR = p(1 + 1/e) \) (see Sec. 2.1). Using this formula in each market, \( p_1(1 + 1/e_1) = p_2(1 + 1/e_2) \), or

\[p_1 = \frac{1 + 1/e_2}{1 + 1/e_1} p_2 \] (4-10)

The demands in both markets must be elastic \( (e < -1) \) (why?). It is apparent that if, say, the elasticity in market 1 (families) is \(-4\) while the elasticity in market 2 (the business travelers) is \(-2\), the price in market 1 will be lower for the families (with these numbers, \( p_1 = (2/3) p_2 \)). Not surprisingly, the monopolist charges a lower price in the market with the higher elasticity.

### PROBLEMS

1. For each of the following functions, find the stationary point and determine whether that point is a relative maximum, minimum, or saddle point of \( f(x_1, x_2) \).
   \( a) \ f(x_1, x_2) = x_1^2 - 4x_1x_2 + 2x_2^2 \)
   \( b) \ f(x_1, x_2) = -4x_1 - 6x_2 + x_1^2 - x_1x_2 + 2x_2^2 \)
   \( c) \ f(x_1, x_2) = 12x_1 - 4x_2 - 2x_1^2 + 2x_1x_2 - x_2^2 \)

2. Using Eq. (4-2), show that the sufficient conditions for \( f(x_1, x_2) \) to achieve a minimum at \( x^0 \) are the relations (4-5).
3. Consider the production function \( y = L^\alpha K^\beta \). Show that this function is strictly concave (downward) for all values of \( L \) and \( K \) if \( 0 < \alpha < 1, 0 < \beta < 1 \) and if \( \alpha + \beta < 1 \). What shape does the function have for \( \alpha + \beta = 1 \)?

4. Show that the production function \( y = \log L^\alpha K^\beta \) is concave for all \( \alpha, \beta > 0 \).

5. Let \( y = f(x_1, x_2) \) and let \( z = F(y) = F(f(x_1, x_2)) = g(x_1, x_2) \). Show that if \( F' > 0 \), then \( g \) has a stationary point at \((x_1^0, x_2^0)\) when and only when \( f \) is stationary there. Under what conditions will \( f \) have a maximum when and only when \( g \) has a maximum?

4.3 AN EXTENDED FOOTNOTE

In the previous section, sufficient conditions for the maximization of a function of two variables were derived via an artifice that reduced the problem to one dimension, or one variable. It is true that if a function has a maximum at some point, then all curves lying in the surface depicted by that function and passing through the maximum point must themselves have a maximum at that point. In that case, therefore, \( y''(t) \leq 0 \) for all such curves. Various plausible-sounding converses of this proposition, however, are not, in general, true. For example, suppose the function \( f(x_1, x_2) \) possesses a maximum when evaluated along all possible polynomial curves, for any values of the coefficients \( a_1, \ldots, a_n, b_1, \ldots, b_n \), for any finite \( n \):

\[
x_1 = x_1^0 + a_1 t + a_2 t^2 + \cdots + a_n t^n
\]

\[
x_2 = x_2^0 + b_1 t + b_2 t^2 + \cdots + b_n t^n
\]

Even if \((x_1(t), x_2(t))\) has a maximum at \( t = 0 \) when evaluated along this wide range of curves, the function \( f(x_1, x_2) \) itself need not have a maximum at \( x_1^0, x_2^0 \).

To illustrate this phenomenon, suppose the curves \((x_1(t), x_2(t))\) are limited to straight lines passing through \((x_1^0, x_2^0)\). That is, consider the curves in the surface \( y = f(x_1, x_2) \) formed by the intersection of that surface and vertical (perpendicular to the \( x_1 x_2 \) plane) planes. Then it is not the case that if all those curves have a maximum, then the function itself has a maximum, as the following counterexample, developed by the mathematician Peano, shows: Consider the function

\[
y = (x_2 - x_1^2)(x_2 - 2x_1^2)
\]

depicted graphically in Fig. 4-1. This function has the value 0 along the curves \( x_2 = x_1^2 \), and along \( x_2 = 2x_1^2 \), both of which are parabolas in the \( x_1 x_2 \) space. In particular, \( y = 0 \) at the origin. The pluses and minuses shown in the diagram reflect the value of the function in the given section of the \( x_1 x_2 \) space. For any point below the lower parabola, \( x_2 < x_1^2 \), and hence \( x_2 < 2x_1^2 \) (the point is also below the upper parabola). Hence \( y \) is the product of two negative numbers and is thus positive. Likewise, above the upper parabola, \( x_2 > 2x_1^2 \); hence \( x_2 > x_1^2 \) and therefore \( y = (+)(+), > 0 \). In between the two parabolas, \( x_2 > x_1^2 \) but \( x_2 < 2x_1^2 \), hence \( y = (+)(-) \leq 0 \). Note how any neighborhood containing the origin possesses both positive and negative values of \( y \). Therefore, the function cannot attain either a maximum or minimum at the origin. That is, since some values are greater than 0 and some less than 0 around the
origin, neither a maximum nor minimum can be achieved there. Rather, something analogous to a saddle point occurs. However, consider the function evaluated along any straight line through the origin, e.g., line \( AA \) in Fig. 4-1. After passing through the upper parabola, the function, along this line, changes from positive to 0 (at the origin) to positive again, implying that the origin is a minimum value of \( y \), evaluated along this or any such line. However, the function itself, as we just have shown, does not have a minimum at the origin. Thus it is not the case that if a function attains a maximum (or minimum) evaluated along all straight lines going through some point that the function necessarily attains a maximum (minimum) there. It is possible to construct functions such that even if \( y(t) \) has a maximum for all polynomial curves in the \( x_1x_2 \) plane, the function itself does not have a maximum.\(^1\) Exactly what class of functions \( x(t) \) for which a valid converse is obtainable seems to be unresolved.

4.4 AN APPLICATION OF MAXIMIZING BEHAVIOR: THE PROFIT-MAXIMIZING FIRM

The tools developed in the previous sections will now be applied to analyze the comparative statics of a profit-maximizing firm that sells its output \( y \) at constant unit price \( p \) and purchases two inputs \( x_1 \) and \( x_2 \) at constant unit factor prices \( w_1 \) and \( w_2 \), respectively. That is, the firm in question is the textbook prototype, facing competitive input and output markets. The production process of the firm will be summarized by the production function, \( y = f(x_1, x_2) \). The production function

will be interpreted here as a technological statement of the maximum output that can be obtained through the combining of two inputs, or factors, $x_1$ and $x_2$. The objective function of this firm is total revenue minus total cost (profits).  

We assert that the firm maximizes this function, i.e.,

$$\text{maximize} \quad \pi = pf(x_1, x_2) - w_1x_1 - w_2x_2 \quad (4-11)$$

The test conditions of this model are the particular values of the input prices $w_1$, $w_2$, and output price $p$. The objective of the model is to be able to state refutable propositions concerning observable behavior, e.g., changes in the levels of inputs used, as the test conditions change, i.e., as factor or output prices change.

The first-order conditions for profit maximization are

$$\pi_1 = \frac{\partial \pi}{\partial x_1} = pf_1 - w_1 = 0 \quad (4-12a)$$

and

$$\pi_2 = \frac{\partial \pi}{\partial x_2} = pf_2 - w_2 = 0 \quad (4-12b)$$

Sufficient conditions for a maximum position are

$$\pi_{11} < 0 \quad \pi_{22} < 0 \quad \text{and} \quad \pi_{11}\pi_{22} - \pi_{12}^2 > 0 \quad (4-13)$$

Since $\pi_{ij} = pf_{ij}$, these second-order conditions reduce to

$$f_{11} < 0 \quad f_{22} < 0 \quad (4-14)$$

and

$$f_{11}f_{22} - f_{12}^2 > 0 \quad (4-15)$$

What is the economic interpretation of these conditions? Equations (4-12) say that a profit-maximizing firm will employ resources up to the point where the marginal contribution of each factor to producing revenues, $pf_i$, the value of the marginal product of factor $i$, is equal to the cost of acquiring additional units of that factor, $w_i$. These are necessarily implied by profit maximization; however, to ensure that the resulting factor employment pertains to maximum rather than minimum profits, conditions (4-14) and (4-15) are needed. Conditions (4-14) are statements of the

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1The student should be wary of the terms firm and profits. With regard to the former, the concept has not been defined here, and there is, in fact, considerable debate in the profession as to exactly what firms are, why they exist at all, and what their boundaries are. With regard to profits, the model leaves unspecified who has claim to the supposed excess of revenues over cost. Alternatively, if $x_1$ and $x_2$ are indeed the only two factors, in whose interest is it to maximize the expression in Eq. (4-11)? In spite of these shortcomings, since the model does yield refutable hypotheses, as we shall see shortly, it is potentially interesting. It might be referred to as a “black box” theory of the firm.
law of diminishing returns. That such a law is involved is easily seen. [Remember, though, conditions (4-13) are sufficient, not necessary—a maximum position is consistent with these relations holding as equalities.] Assuming it was worthwhile to hire one unit of that factor in the first place, if the value of the marginal product of that factor was increasing, the firm would hire that factor without bound, since the input would be generating more income than it was getting paid. Hence a finite maximum position is inconsistent with increasing marginal productivity.

However, diminishing marginal productivity in each factor does not, by itself, guarantee that a maximum profit position will be achieved. Condition (4-15) is also required. This relation, though less intuitive than diminishing marginal productivity, arises from the fact that changes in one factor affect the marginal products of the other factors as well as its own marginal product, and the overall effect on all marginal products must be akin to diminishing marginal productivity. Suppose, for example, that \( f_{12} = f_{21} = \partial MP_1 / \partial x_2 = \partial MP_2 / \partial x_1 \) is very large, in absolute terms, relative to \( f_{11} = \partial MP_1 / \partial x_1 \) and \( f_{22} = \partial MP_2 / \partial x_2 \). That is, suppose a change in \( x_1 \), say, affects the marginal product of factor 2 much more than the marginal product of factor 1. Then consider the consequences of an increase of one unit of \( x_1 \). In Fig. 4-2, if \( f_{12} = f_{21} > 0 \), \( MP_1 \) initially declines; however \( MP_2 \) shifts upward by a considerable amount, causing the firm to purchase many additional units of \( x_2 \). However, these additional units of \( x_2 \) have an effect on \( MP_1 \). Since \( f_{12} = \partial MP_1 / \partial x_2 > 0 \), \( MP_1 \) also shifts up, by a relatively large amount. The final result, then, is that an increase in \( x_1 \) can lead to an increase in \( MP_1 \), if the cross-effects are large enough. Hence the original factor employment levels, though characterized by diminishing marginal productivity in each factor, do not nonetheless describe a profit maximum position, since it is clearly profitable in this case to increase the usage of both \( x_1 \) and \( x_2 \) together. In the case where \( f_{12} \) is negative and large relative to \( f_{11} \) and \( f_{22} \), the analysis is similar. An increase in \( x_1 \) causes a relatively large fall in \( MP_2 \), a fall

![Figure 4-2](image)

**FIGURE 4-2**

The Law of Diminishing Returns. The fact that \( f_{11} < 0, f_{22} < 0 \) alone is not sufficient to guarantee a finite profit-maximum position. The cross-effects between the two factors must be considered. If \( x_1 \) is increased, \( MP_1 \) might shift out, say, a great deal, shifting \( MP_1 \) out resulting in a net increase in \( MP_1 \), even though \( f_{11} = \partial MP_1 / \partial x_1 < 0 \). This will occur if \( f_{12} \) is large. Hence the condition \( f_{11} f_{22} - f_{12}^2 \geq 0 \) is also needed to achieve a finite profit maximum.
seen. [Remember, \( x_2 \) is the marginal position is it was worthwhile marginal product thought bound, since said. Hence a finite activity.

**Does not**, by itself, iteration (4-15) is also original productivity, resulting from the other act on all marginal pose, for example, the terms, relative to \( x_1 \), say, affects product of factor \( x_1 \). In Fig. 4-2, if by a considerable \( x_2 > 0 \), \( MP_1 \) also has an increase in enough. Hence the diminishing marginal maximum position, of both \( x_1 \) and \( x_2 \) \( f_1 \) and \( f_2 \), the all in \( MP_2 \), a fall in \( x_2 \) and hence a relatively large increase in \( MP_1 \). (Remember, \( \partial MP_1 / \partial x_2 < 0 ! \))

In this case, increasing one factor and decreasing the other (together) will increase profits.

Let us return now to the marginal relations (4-12). The purpose of formulating this model is not simply to assert the implied marginal reasoning; that is a rather sterile endeavor. The purpose of this analysis is to be able to formulate refutable hypotheses as to how firms react to changes in the parameters they face; in particular, in this case, to changes in factor and output prices. To this end, we now consider the comparative statics of this model.

The first-order conditions in complete form are

\[
pf_1(x_1, x_2) - w_1 = 0
\]

\[
pf_2(x_1, x_2) - w_2 = 0
\]

These are two implicit relations in essentially five unknowns: \( x_1, x_2, w_1, w_2, \) and \( p \). Under the "right" conditions (to be discussed in what follows) it is possible to solve for two of these values in terms of the other three. In particular, we can solve for the choice functions

\[
x_1 = x_1^*(w_1, w_2, p)
\]

(4-16a)

and

\[
x_2 = x_2^*(w_1, w_2, p)
\]

(4-16b)

Equations (4-16) represent the factor demand curves. These relations indicate the amount of each factor that will be hired, according to this model, as a function of the factor prices and product price; they are the choice functions of this model. Assuming that it is possible to solve for Eqs. (4-16), it becomes meaningful to ask questions regarding the signs of the following six partial derivatives, which comprise the comparative statics of the profit-maximizing model:

\[
\frac{\partial x_1^*}{\partial w_1} \quad \frac{\partial x_1^*}{\partial w_2} \quad \frac{\partial x_1^*}{\partial p} \quad \frac{\partial x_2^*}{\partial w_1} \quad \frac{\partial x_2^*}{\partial w_2} \quad \frac{\partial x_2^*}{\partial p}
\]

(4-17)

These partials indicate the marginal changes in factor employment due to given price changes. It is important to keep in mind that in order to write down these relations and interpret them in some meaningful fashion, the explicit functions \( x_1^* \) must be well defined. Also note that the preceding factor demand curves are not the marginal product curves. The marginal product functions \( f_1 \) and \( f_2 \) are expressed in terms of the factor inputs, while the factor demand curves are expressed in terms of prices, and dependent upon the behavioral assertion of the model.

Substituting Eqs. (4-16) back into Eqs. (4-12) produces the following identities:

\[
pf_1(x_1^*(w_1, w_2, p), x_2^*(w_1, w_2, p)) - w_1 \equiv 0
\]

(4-18a)

and

\[
pf_2(x_1^*(w_1, w_2, p), x_2^*(w_1, w_2, p)) - w_2 \equiv 0
\]

(4-18b)
Recall the monopolist tax example of Chap. 1, where the solution \( x = x^*(t) \) of the first-order relation (which set marginal revenue equal to marginal cost plus the tax) was then substituted back into that relation, yielding an identity in the tax rate \( t \). For the same reasons, the relations (4-18) are identities in the prices \( w_1, w_2, \) and \( p \). The factor demand functions \( x_1^* \) and \( x_2^* \) are precisely those levels of \( x_1 \) and \( x_2 \) that the entrepreneur employs to keep the value of the marginal products of each factor equal to the wage of each factor, for any prices.

Hence, the assertion that the firm always obeys Eqs. (4-12), for any prices, converts those equations to the identities (4-18). Being identities, the relations (4-18) can be differentiated implicitly with respect to the various prices, producing relations that allow solutions for the partial derivatives (4-17). The general procedure is exactly the same as in the monopolist example. However, in this example, two first-order relations are present instead of only one, and that fact makes the algebra more difficult.

Before we do the differentiation, note that if the firm's production functions were in fact known, then one could actually solve for the factor demand curves explicitly. In that case we could know the total quantities involved in this model, a happy state of affairs. The factor demand curves (4-16) could be differentiated directly to yield the partial derivatives (4-17). However, the economist is not likely to have this much information. Nonetheless, it is still possible to state refutable hypotheses concerning marginal quantities, through implicit differentiation of the identities (4-18).

Differentiating (4-18a) and (4-18b) partially, with respect to \( w_1 \), using the chain rule (remembering that \( f_1 \) is a function of \( x_1 \) and \( x_2 \), which are in turn functions of \( w_1, w_2, \) and \( p \), etc.),

\[
p \frac{\partial f_1}{\partial x_1} \frac{\partial x_1^*}{\partial w_1} + p \frac{\partial f_1}{\partial x_2} \frac{\partial x_2^*}{\partial w_1} - 1 = 0
\]

\[
p \frac{\partial f_2}{\partial x_1} \frac{\partial x_1^*}{\partial w_1} + p \frac{\partial f_2}{\partial x_2} \frac{\partial x_2^*}{\partial w_1} = 0
\]

Using subscript notation, these can be written

\[
p f_{11} \frac{\partial x_1^*}{\partial w_1} + p f_{12} \frac{\partial x_2^*}{\partial w_1} = 1 \quad (4-19a)
\]

\[
p f_{21} \frac{\partial x_1^*}{\partial w_1} + p f_{22} \frac{\partial x_2^*}{\partial w_1} = 0 \quad (4-19b)
\]

Although the identities (4-19) look complicated, they are a good deal simpler in form than (4-18). Whereas the first-order relations (4-18) are in general complicated algebraic expressions, (4-19a) and (4-19b) are simple linear relations in the unknowns \( \partial x_1^*/\partial w_1 \) and \( \partial x_2^*/\partial w_1 \). That is, (4-19a) and (4-19b) are of the same form as the elementary system of two simultaneous linear equations in two unknowns. The coefficients of the unknowns are the functions \( p f_{11}, p f_{12}, \) etc., but the system is
tion \( x = x^*(t) \) of the final cost plus the tax \( y \) in the tax rate \( t \). For \( \lambda w_1, w_2, \) and \( p \). The of \( x_1 \) and \( x_2 \) that the s of each factor equal 

\[ 12 \], for any prices, s, the relations (4-18) is, producing relations procedure is exactly nple, two first-order es the algebra more 

production functions ctor demand curves olved in this model, dld be differentiated monist is not likely le to state refutable differentiation of the o \( w_1 \), using the chain in turn functions of 

still simple in that no products, or squares, of the terms \( \partial x^*_1 / \partial w_1 \), etc., are involved. And this is fortunate, since the goal of this analysis is to solve for those terms, i.e., find expressions for the partials of the form \( \partial x^*_1 / \partial w_j \).

To solve for \( \partial x^*_1 / \partial w_1 \), for example, multiply (4-19a) by \( f_{22} \) and (4-19b) by \( f_{12} \) and subtract (4-19b) from (4-19a). This yields, after some factoring (remember that \( f_{12} = f_{21} \)),

\[
p(f_{11}f_{22} - f_{12}^2) \frac{\partial x^*_1}{\partial w_1} = f_{22}
\]

Now, if \( f_{11}f_{22} - f_{12}^2 \neq 0 \), that term can be divided on both sides, yielding

\[
\frac{\partial x^*_1}{\partial w_1} = \frac{f_{22}}{p(f_{11}f_{22} - f_{12}^2)}
\]

(4-20a)

In like fashion, one obtains

\[
\frac{\partial x^*_2}{\partial w_1} = \frac{-f_{11}}{p(f_{11}f_{22} - f_{12}^2)}
\]

(4-20b)

To obtain the responses of the firm to changes in \( w_2 \), differentiate Eqs. (4-18) with respect to \( w_2 \). Noting that \( w_2 \) enters only the second equation explicitly, the system of comparative statics relations becomes

\[
pf_{11} \frac{\partial x^*_1}{\partial w_2} + pf_{12} \frac{\partial x^*_2}{\partial w_2} = 0
\]

\[
pf_{21} \frac{\partial x^*_1}{\partial w_2} + pf_{22} \frac{\partial x^*_2}{\partial w_2} = 1
\]

Solving these equations as before yields

\[
\frac{\partial x^*_1}{\partial w_2} = \frac{-f_{12}}{p(f_{11}f_{22} - f_{12}^2)}
\]

(4-20c)

\[
\frac{\partial x^*_2}{\partial w_2} = \frac{f_{11}}{p(f_{11}f_{22} - f_{12}^2)}
\]

(4-20d)

Note that sufficient condition (4-15), \( f_{11}f_{22} - f_{12}^2 > 0 \), is enough to guarantee \( f_{11}f_{22} - f_{12}^2 \neq 0 \) and hence allow solution for these partials (4-20a–d). This is not mere coincidence; it is in fact an application of the "implicit function theorem" in mathematics that will be dealt with more generally in Chap. 5. The condition \( f_{11}f_{22} - f_{12}^2 \neq 0 \) is precisely the mathematical condition to allow solution (locally, not everywhere) for the factor demand curves \( x^*_1(w_1, w_2, p) \) in the first place. The relevance of that term is brought out in the situation for the partial derivatives.

\[ \text{[In accordance with general custom, we will use the equality rather than the identity sign when the special emphasis is not required.} \]
What refutable hypotheses emerge from this analysis? Condition (4-15) implies that the denominators of (4-20a–d) are all positive. Condition (4-14), \( f_{11}, f_{22} < 0 \), (diminishing marginal productivity) makes the numerators of (4-20a) and (4-20d) negative. Hence, the regular (sufficient) conditions for maximum profits imply that the factor demand curves must be downward-sloping in their respective factor prices. The model implies that changes in a factor price will result in a change in the usage of that factor in the opposite direction.

What about the cross-effects \( \delta x_i^*/\delta w_2, \delta x_j^*/\delta w_1 \)? The most remarkable aspect of these two expressions is that they are always equal, by inspection of (4-20b) and (4-20c), noting that \( f_{12} = f_{21} \). This reciprocity relation,

\[
\frac{\delta x^*_i}{\delta w_2} = \frac{\delta x^*_j}{\delta w_1}
\]

is representative of a number of such relations that appear in economics, as well as in the physical sciences, when maximizing principles are involved. As is obvious from the forms of these expressions, however, the reciprocity relations are no less intuitive than the mathematical theorem from which they originate—the invariance of cross-partial derivations to the order of differentiation.

Beyond the equality of these cross-effects, there is little else to say about them. The sign of \( f_{12} \) is not implied by the maximization hypothesis; hence the sign of \( \delta x_i^*/\delta w_j, i \neq j \) is similarly not implied. No refutable proposition emerges about these terms from the profit maximization model. All observed events relating, say, to the change in labor employment when the rental rate on capital increases are consistent with the previous model.

Suppose now it is desired to find expressions relating to the effects of changes in the output price \( p \). The procedure here is identical up through relations (4-18). Then, we differentiate those identities partially with respect to \( p \), producing

\[
p f_{11} \frac{\delta x^*_i}{\delta p} + p f_{12} \frac{\delta x^*_j}{\delta p} = -f_1 \tag{4-21a}
\]

\[
p f_{21} \frac{\delta x^*_i}{\delta p} + p f_{22} \frac{\delta x^*_j}{\delta p} = -f_2 \tag{4-21b}
\]

remembering that the product rule is called for in differentiating the terms \( pf_1, pf_2 \).

Solving these equations for \( \delta x^*_i/\delta p \) and \( \delta x^*_j/\delta p \) yields

\[
\frac{\delta x^*_i}{\delta p} = \frac{-f_1 f_{22} + f_3 f_{12}}{p (f_{11} f_{22} - f_{12}^2)} \tag{4-22a}
\]

\[
\frac{\delta x^*_j}{\delta p} = \frac{-f_2 f_{11} + f_3 f_{12}}{p (f_{11} f_{22} - f_{12}^2)} \tag{4-22b}
\]

It can be seen that no refutable implications emerge from these expressions. An increase in output price can lead to an increase or a decrease in the use of either factor, since the sign of \( f_{12} \) is unknown. (Note that if \( f_{12} > 0 \) is assumed, \( \delta x^*_i/\delta p > 0 \) and \( \delta x^*_j/\delta p > 0 \).) It is possible to show, however, that it cannot be the case that both
\[ \frac{\partial x_1^*}{\partial p} < 0 \text{ and } \frac{\partial x_2^*}{\partial p} < 0 \] simultaneously. An increase in output price cannot lead to less use of both factors. The proof of this is left as an exercise.

**The Supply Function**

It is also possible to ask how output varies when a parameter changes. Since \( y = f(x_1, x_2) \),

\[ y^* = f(x_1^*, x_2^*) \]

where \( y^* \) is the profit-maximizing level of output.

The factor demand curves are functions of the prices,

\[ x_i = x_i^*(w_1, w_2, p) \quad i = 1, 2 \]

Substituting these functions into \( f(x_1^*, x_2^*) \) yields

\[ y^* = f(x_1^*(w_1, w_2, p), x_2^*(w_1, w_2, p)) = y^*(w_1, w_2, p) \quad (4-23) \]

Equation (4-23) represents the supply function of this firm. It shows how output is related (1) to output price \( p \), and (2) to the factor prices. Though the supply curve is commonly drawn only against output price \( p \), factor prices must also enter the function, since factor costs obviously affect the level of output a firm will choose to produce.

How will output be affected by an increase in output price? To answer this, differentiate (4-23) with respect to \( p \) using the chain rule,

\[ \frac{\partial y^*}{\partial p} = \frac{\partial f}{\partial x_1} \frac{\partial x_1^*}{\partial p} + \frac{\partial f}{\partial x_2} \frac{\partial x_2^*}{\partial p} \]

or

\[ \frac{\partial y^*}{\partial p} = f_1 \frac{\partial x_1^*}{\partial p} + f_2 \frac{\partial x_2^*}{\partial p} \quad (4-24) \]

Now, substitute Eqs. (4-22) into this expression. This yields

\[ \frac{\partial y^*}{\partial p} = \frac{-f_1 f_{22} + 2f_{12} f_1 f_2 - f_2^2 f_{11}}{p (f_{11} f_{22} - f_{12}^2)} \quad (4-25) \]

The denominator of this expression is positive by the sufficient second-order conditions. We also can infer, from Eq. (4-7), that the numerator is also positive. Therefore,

\[ \frac{\partial y^*}{\partial p} > 0 \quad (4-26) \]

This says that the sufficient second-order conditions for profit maximization imply that the supply curve, as usually drawn, must be upward-sloping. It also provides an explanation as to why it cannot be the case that both \( \frac{\partial x_1^*}{\partial p} \) and \( \frac{\partial x_2^*}{\partial p} \) are negative. If \( p \) increases, output will increase. It is impossible, with positive marginal products, to produce more output with less of both factors.
It is also possible to derive some reciprocity relationships with regard to the output supply and factor demand functions. In particular, one can show

$$\frac{\partial y^*}{\partial w_i} = -\frac{\partial x^*_i}{\partial p} \quad i = 1, 2$$

(4-27)

The signs of these expressions are indeterminate; however, this curious reciprocity result is valid. Its proof is left as an exercise.

The tools used in this analysis include the solution of simultaneous linear equations. For this reason, the next chapter is on the theory of matrices and determinants. It will be of great advantage to be able to have a general way of expressing the solutions of such equation systems, instead of laboriously working through each expression separately.

### 4.5 Homogeneity of the Demand and Supply Functions; Elasticities

Suppose the economy were to experience a perfectly neutral inflation, i.e., input and output prices all increasing in the same proportion, say 10 percent. Since relative prices would not have changed, it would be important that the model predict that no decisions would be changed in response to this. In other words, the factor demand functions and the supply function should be homogeneous of degree 0 in all prices. Is this the case?

The factor demand functions are the simultaneous solutions to the first-order conditions

$$pf_1(x_1, x_2) - w_1 = 0$$
$$pf_2(x_1, x_2) - w_2 = 0$$

Suppose $w_1$, $w_2$, and $p$ all change in the same proportion, i.e., these prices become $tw_1$, $tw_2$, and $tp$, where $t$ is some scalar factor. The factor demand functions are now evaluated at these new prices: $x^*_1(tw_1, tw_2, tp)$, $x^*_2(tw_1, tw_2, tp)$. These functions are the solutions to the first-order equations at the new prices:

$$(tp)f_1(x_1, x_2) - (tw_1) = 0$$
$$(tp)f_2(x_1, x_2) - (tw_2) = 0$$

But these equations are clearly equivalent to the original ones; all that has happened algebraically is that the equations have been multiplied through by $t$. Since the equations from which the two solutions are derived are algebraically identical, the solutions must also be identical. That is,

$$x^*_i(tw_1, tw_2, tp) = x^*_i(w_1, w_2, p) \quad i = 1, 2$$

In this model, therefore, the factor-demand functions are necessarily homogeneous of degree 0. [It quickly follows that the supply function $y^*(w_1, w_2, p)$ must also be homogeneous of degree 0; its proof is left as an exercise.]
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Notice that the preceding proof in no way depends on any assumption about
the functional form of the production function. In particular, to head off a frequently
made error, it is not the case that the production function must be homogeneous of
some degree. The demand functions are not the partial derivatives of the production
function. They are the simultaneous solutions to the first-order equations. The result
follows because those first-order equations are linear in \( w_1, w_2, \) and \( p \). When each
of those parameters is increased in the same proportion, the factor of proportionality
cancels out of the first-order equations, leaving the system unchanged.

**Elasticities**

The properties of the factor demand functions \( x_i^*(w_1, w_2, p) \) are often stated in terms
of dimensionless elasticity expressions instead of the slopes (partial derivatives).
These elasticities are defined as

\[ \epsilon_{ij} = \lim_{\Delta w_j \to 0} \frac{\Delta x_i/x_i}{\Delta w_j/w_j} \]  

(4-28)

The elasticity \( \epsilon_{ij} \) represents the (limit of the) percentage change in the use of factor
\( x_i \) per percentage change in price of factor \( j \). When \( i = j \), this is called the own
elasticity of factor demand; when \( i \neq j \), it is called a cross-elasticity.

Taking limits and simplifying the compound fraction,

\[ \epsilon_{ij} = \frac{w_j \partial x_i^*}{x_i^* \partial w_j} \]  

(4-29)

This is the definition we shall use throughout. In like fashion, one can define the
output price elasticity of factor demand as the percentage change in the utilization
of a factor per percentage change in output price (holding factor prices constant), or

\[ \epsilon_p = \lim_{\Delta p \to 0} \frac{\Delta x_i/x_i}{\Delta p/p} = \frac{p \partial x_i^*}{x_i^* \partial p} \]  

(4-30)

Elasticities are dimensionless expressions, as can be seen by inspection: the
units all cancel. To a mathematician, they are logarithmic derivatives. For example,
letting \( u_i = \log x_i, v_j = \log w_j \),

\[ \frac{du_i}{dv_j} = \frac{dx_i/x_i}{dw_j/w_j} = \frac{w_j dx_i}{x_i dw_j} \]

The notation changes appropriately for partial derivatives. Many economists prefer
to deal with elasticities; others prefer the slopes (unadorned partial derivatives). It
is mainly a matter of taste.

By applying Euler's theorem to the factor demand functions \( (x_i, p, w_1, w_2) \) the example
that follows, we can derive some relationships concerning the elasticities and
cross-elasticities of demand:
\[
\left( \frac{\partial x_1^*}{\partial w_1} \right) w_1 + \left( \frac{\partial x_1^*}{\partial w_2} \right) w_2 + \left( \frac{\partial x_1^*}{\partial p} \right) p = 0
\]

Dividing through by \( x_1 \) yields
\[
\epsilon_{11} + \epsilon_{12} + \epsilon_{1p} \equiv 0
\]
with a similar expression holding for \( x_2 \). In general, for models with \( n \) factors of production,
\[
\sum_j \epsilon_{ij} + \epsilon_{ip} \equiv 0 \quad i = 1, \ldots, n
\] (4-31)

4.6 THE LONG RUN AND THE SHORT RUN: AN EXAMPLE OF THE LE CHÂTELIER PRINCIPLE

It is commonplace to assert that certain factors of production are "fixed" over certain time intervals, e.g., that capital inputs cannot be varied over the short run. In fact, of course, these statements are incorrect; virtually anything can be changed, even quickly, if the benefits of doing so are great enough. Yet it does seem that certain inputs are more easily varied, i.e., less costly to vary than others. The extreme abstraction of this is to simply assert that for all intents and purposes, one factor is fixed. (A government edict fixing some level of input would suffice, if ignoring such edict carried with it a sufficiently long jail sentence.) How would a profit-maximizing firm react to changes in the wage of one factor \( x_1 \), when it found that it could not vary the level of \( x_2 \) employed? Would the factor demand curve for \( x_1 \) be more elastic or less elastic than previously?

Suppose \( x_2 \) is held fixed at \( x_2 = x_2^0 \). The profit function then becomes
\[
\max \pi = pf(x_1, x_2^0) - w_1 x_1 - w_2 x_2^0.
\]

In this case, there is only one decision variable: \( x_1 \). Hence the first-order condition for maximization is simply
\[
\pi_1 = pf_1(x_1, x_2^0) - w_1 = 0
\] (4-32)
and the sufficient second-order condition is
\[
\pi_{11} = pf_{11} < 0
\] (4-33)
We are dealing with a one-variable problem with, now, four parameters, \( w_1, w_2, p, \) and \( x_2^0 \). The factor demand curve, obtained from Eq. (4-32), is
\[
x_1 = x_1^* (w_1, p, x_2^0)
\] (4-34)
where \( x_1^* \) stands for short-run demand. Note, however, that \( w_2 \) does not enter this factor demand curve. With \( x_2 \) fixed, \( w_2 x_2^2 \) is a fixed cost, and thus \( w_2 \) is irrelevant for the choice of \( x_1 \) in the short run. The slope of the short-run factor demand curve
is $\partial x_1^f / \partial w_1$. To obtain an expression for this partial, substitute, as before, $x_1^l$ into Eq. (4-32), yielding the identity

$$p f_1(x_1^l, x_2^0) - w_1 = 0$$

Differentiating this identity with respect to $w_1$ yields

$$p f_{11} \frac{\partial x_1^f}{\partial w_1} \equiv 1$$

or

$$\frac{\partial x_1^f}{\partial w_1} \equiv \frac{1}{p f_{11}} < 0$$  \hspace{1cm} (4-35)

Thus, the short-run factor demand curve is downward-sloping. How does this slope compare with $\partial x_1^L / \partial w_1 = \partial x_1^f / \partial w_1$ ($x_1^f$ for long-run demand) derived in (4-20a)? Taking the difference,

$$\frac{\partial x_1^L}{\partial w_1} - \frac{\partial x_1^f}{\partial w_1} = \frac{f_{22}}{p (f_{11} f_{22} - f_{12}^2)} - \frac{1}{p f_{11}}$$

Combining terms yields

$$\frac{\partial x_1^L}{\partial w_1} - \frac{\partial x_1^f}{\partial w_1} = \frac{f_{12}^2}{p f_{11} (f_{11} f_{22} - f_{12}^2)} < 0$$  \hspace{1cm} (4-36)

a determinately negative expression due to the second-order conditions (4-15) and (4-33). Since both $\partial x_1^L / \partial w_1$ and $\partial x_1^f / \partial w_1$ are negative, (4-36) says that the change in $x_1$ due to a change in its price is larger, in absolute value, when $x_2$ is variable (the long run) than when $x_2$ is fixed (the short run). This result is sometimes referred to as the second law of demand. It is in agreement with intuition—if the price of labor, say, were to increase relative to capital’s price, the firm would attempt to substitute out of labor. The degree to which it could do this, however, would be impaired if it could not at the same time increase the amount of capital employed. Hence the model implies that over longer periods of time, as the other factor becomes “unstuck,” the demand for the less-costly-to-change factor will become more elastic. Incidentally, the usual factor demand diagrams are drawn with the dependent variable $x_1$ on the horizontal axis; in that case the long-run factor demand curves appear flatter than the short-run curves. Also note that this comparison makes sense only if the level of $x_2$ employed is the same in both cases. That is, the preceding is a local theorem, holding only at the point where the short- and long-run demand curves intersect, i.e., at the common values of $x_2$. At any finite distance from this intersection, the long-run demand curve might actually be less elastic than the short-run curve.

The result contained in this section is commonly believed to be empirically true, simply as a matter of assertion. It is interesting and noteworthy that this type of behavior is in fact mathematically implied by a maximization hypothesis. These types of relations are sometimes referred to as Le Châtelier effects, after the similar tendency of thermodynamic systems to exhibit the same types of responses. Some
generalizations of this phenomenon and its relation to "envelope" theorems will be presented in Chap. 7.

A More Fundamental Look at the Le Châtelier Principle

Although the above algebra proves that when the level of one factor, say, \( x_2 \), is held fixed at its profit-maximizing level, the resulting short-run factor demand curve is less elastic than the long-run curve at that point, the proof provides no insight into the fundamental relationship between the long- and short-run factor demands. If a consistent relationship exists between the partial derivatives of two separate demand functions, it must be the case that some fundamental identity exists that relates the two demands to each other.

In the instant case, consider what would convert the short-run demand to the long-run demand. We would accomplish this by letting \( x_2 \) adjust to the change in \( w_1 \) instead of holding it fixed. In fact, we can define the long-run factor demand in terms of the short-run demand by letting \( x_2 \) (the "fixed" factor) adjust to its profit-maximizing levels as \( w_1 \) changes:

\[
x_1^*(w_1, w_2, p) = x_1^*(w_1, p, x_2^*(w_1, w_2, p))
\]

(4-37)

This identity is the fundamental relationship between the short- and long-run factor demands. Using this identity, we can demonstrate and explain the Le Châtelier results with much greater clarity. The right-hand side of Eq. (4-37) is known as a conditional demand.\(^1\)

The relation (4-37) is an identity; it holds for all \( w_1, w_2, \) and \( p \). We can therefore validly differentiate it with respect to any of those arguments. In particular, differentiate with respect to \( w_1 \), noting that on the right-hand side of (4-37), \( w_1 \) enters once explicitly by itself, and another time as an argument of \( x_2^* \):

\[
\frac{\partial x_1^*}{\partial w_1} = \frac{\partial x_1^*}{\partial w_1} + \left( \frac{\partial x_1^*}{\partial x_2^*} \right) \left( \frac{\partial x_2^*}{\partial w_1} \right)
\]

(4-38)

Inspect the notation in the chain rule part of the right-hand side of (4-38) carefully: \( x_1^* \) is a function of \( x_2^* \) (not \( x_2^* \)); the functional dependence of \( x_2 \) on \( w_2 \) is defined by the long-run demand \( x_2^* \).

Equation (4-38) reveals that the slopes of the short- and long-run factor demand functions differ by a term representing the product of two effects: the change in \( x_2 \) resulting from a change in \( w_1 \), and the change in \( x_1 \) that would be induced by a (parametric) change in \( x_2 \). This product is easily seen to represent the marginal

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\(^1\)This approach was first developed by Robert Pollak, for the case of consumer demands. See his "Conditional Demand Functions and Consumption Theory," *Quarterly Journal of Economics, 83*: 60–78, February 1969.
theorems will be

effect on \( x_1 \) of allowing \( x_2 \) to vary as \( w_1 \) changes. The important question is, can this latter term be signed?

It should seem plausible that \( \partial x_1^*/\partial x_2^* \) and \( \partial x_2^*/\partial w_1 \) have opposite signs. From

reciprocity, \( \partial x_1^*/\partial w_1 = \partial x_1^*/\partial w_2 \). Increasing \( x_2^* \) parametrically accomplishes di-

rectly what a decrease in \( w_2 \) would induce. We can verify this algebraically as follows. Differentiating (4-37) with respect to \( w_2 \)

\[
\frac{\partial x_1^*}{\partial w_2} = \left( \frac{\partial x_1^*}{\partial x_2^*} \right) \left( \frac{\partial x_2^*}{\partial w_2} \right)
\]  

(4-39)

Since \( \partial x_2^*/\partial w_2 < 0 \), \( \partial x_1^*/\partial w_2 \) and \( \partial x_1^*/\partial x_2^* \) are of opposite sign. Using Eq. (4-39)

to eliminate \( \partial x_1^*/\partial x_2^* \) from Eq. (4-38), and using reciprocity,

\[
\frac{\partial x_1^*}{\partial w_1} = \frac{\partial x_1^*}{\partial w_1} + \left( \frac{\partial x_1^*/\partial w_1}{\partial x_2^*/\partial w_2} \right)^2
\]  

(4-40)

Since the last term must be negative, Eq. (4-40) says that \( \partial x_1^*/\partial w_1 \) is more nega-

tive than \( \partial x_1^*/\partial w_1 \), the Le Châtelier result. More importantly, it illuminates the

fundamental relationship between the long- and short-run factor demand functions.

A similar analysis can be used to show that the long-run output supply function is more elastic than the short-run function. The fundamental identity is

\[ y^*(w_1, w_2, p) = y^*(w_1, p, x_2^*(w_1, w_2, p)) \]  

(4-41)

Differentiating with respect to \( p \),

\[
\frac{\partial y^*}{\partial p} = \frac{\partial y^*}{\partial c} + \left( \frac{\partial y^*}{\partial x_2^*} \right) \left( \frac{\partial x_2^*}{\partial p} \right)
\]  

(4-42)

By differentiating (4-41) with respect to \( w_2 \) and using a reciprocity condition, it can be shown that \( \partial y^*/\partial p > \partial y^*/\partial p \). The proof is left as an exercise.

We shall employ this technique throughout this book. In so doing, many expressions that were once difficult to prove become transparently simple.

To sum up, it has again been possible to state refutable propositions about some marginal quantities, in spite of the scarcity of information contained in the model. Should further information be used, e.g., the specific functional form of the production function, or, less grandiosely, independent measures of the sign of the cross-effect \( f_{12} \), additional restrictions can be placed on the signs of the partial derivatives of the factor demand functions.

**PROBLEMS**

1. Show that no refutable implications emerge from the profit maximization model with regard to the effects of changes in output price on factor inputs. Show, however, that it cannot be the case that both factors decrease when output price is increased.

2. Show that the rate of change of output with respect to a factor price change is equal to the negative of the rate of change of that factor with respect to output price, i.e., Eq. (4-27).
3. (Very messy, but you should probably do this once in your life.) Consider the production function \( y = x_1^r x_2^s \). Find the factor demand curves and the comparative statics of a profit-maximizing firm with this production function. Be sure to review Prob. 3, Sec. 4.2, first. Show that for this firm, the sign of the cross-effect term, \( \partial x_1^r / \partial x_2^s \), is negative.

4. There are several definitions of complementary and substitute factors in the literature, among which are:
   (i) "Factor 1 is a substitute (complement) for factor 2 if the marginal product of factor 1 decreases (increases) as factor 2 is increased."
   (ii) "Factor 1 is a substitute (complement) for factor 2 if the quantity of factor 1 employed increases when the price of factor 2 increases (decreases)."
   (a) Show that both of these definitions are symmetric; i.e., if factor 1 is a substitute for factor 2, then factor 2 can't be a complement to factor 1.
   (b) Show that these two definitions are equivalent in the two-factor, profit maximization model.
   (c) Do you think that these two definitions will be equivalent in a model with three or more factors? Why?

5. Consider again Example 3, Sec. 4.2, wherein a monopolist sells his or her output in two separate markets. Suppose a per-unit tax \( t \) is placed on output sold in the first market.
   (a) Show that an increase in \( t \) will reduce the output sold in market 1.
   (b) What does the maximization hypothesis alone imply about the response of output in the second market to an increase in \( t \)?
   (c) Show that it is possible that an increase in the tax on market 1 can lead to an increase in total output \( x^*(t) = x_1^r(t) + x_2^s(t) \), even assuming the usual sufficient second-order conditions. Under what circumstances (slopes of the marginal cost and marginal revenue functions) does this occur? (This possibility is known as the Hotelling taxation paradox, after Harold Hotelling, an early pioneer of modern economics and statistics, who first explored it.)
   (d) Suppose the output in market 2 were held fixed at the previously profit-maximizing level, by government regulation. Show that the response in output in market 1 to a tax increase is less in absolute terms in the regulated situation than in the unregulated situation. Provide an intuitive explanation for this.

6. The Le Châtelier results of Sec. 4.4 (also Prob. 5) hold, regardless of whether the two factors are complementary or substitutes. Explain the phenomenon intuitively for the case of complementary factors.

7. A monopolist sells his or her output in two markets, with revenue functions \( R_1(y_1), R_2(y_2) \), respectively. Total cost is a function of total output, \( y = y_1 + y_2 \). The same per-unit tax, \( t \), is levied on output sold in both markets.
   (a) Find \( \partial y_1^* / \partial t, \partial y_2^* / \partial t \), and \( \partial y^* / \partial t \), where \( y_1^* \) is the profit-maximizing level of output in market 1 and \( y^* = y_1^* + y_2^* \). Which, if any, of these partials have a sign implied by profit maximization?
   (b) Suppose output \( y_2 \) is held fixed. Find \( (dy_1^* / dt) y_2 \). Does \( (dy^* / dt) y_2 \) have a determinate sign?

8. Consider the following two models of a discriminating monopolist subject to a tax in one market:
   (i) \( \max R_1(x_1) + R_2(x_2) - C(x_1 + x_2) - tx_1 \)
   (ii) \( \max R(x_1, x_2) - C(x_1, x_2) - tx_1 \)
   In model (i), cost is a function only of total output, whereas in (ii), cost and revenue are more complicated (and general) functions of both outputs. The tax rate \( t \) is a parameter. What are the observable similarities and differences between these two models?
 Consider a profit-maximizing firm with the production function \( y = f(x_1, x_2) \), facing output price \( p \) and factor prices \( w_1 \) and \( w_2 \). Suppose this firm is taxed according to the total cost of factor 2, i.e., tax = \( t w_2 x_2 \).

(a) Derive the factor demand functions; i.e., show where they come from, etc. Are these choice functions homogeneous of any degree in any of the parameters?

(b) Show that if the tax rate rises, the firm will use less of factor 2.

(c) Show that \( \frac{\partial x_2^*}{\partial t} = w_2 \frac{\partial x_2^*}{\partial w_1} \).

(d) Suppose that factor 1 is held fixed at its profit-maximizing level. Show that the response of factor 2 to a change in the tax rate is less in absolute value than before.

10. Consider a monopolistic firm that hires two inputs \( x_1 \) and \( x_2 \) in competitive factor markets at wages \( w_1 \) and \( w_2 \), respectively. The firm's revenue function is expressible in terms of the inputs as \( R(x_1, x_2) \). Assuming profit maximization,

(a) Indicate the derivation of the factor demand functions. Are these factor demands homogeneous of some degree in wages?

(b) Show that the factor demand curves are downward-sloping in their own prices.

(c) Is a refutable hypothesis forthcoming as to how the total revenue of this firm would change with regard to a change in a factor price?

11. Consider a profit-maximizing U.S. monopolistic firm that produces some good \( y \) at two different plants, with (total) cost functions \( C_1(y_1), C_2(y_2) \). The total revenue function of this firm is \( R(y) \), where \( y = y_1 + y_2 \). Plant 2 is located in Canada, and output from that plant is subject to a U.S. tariff (tax) in the amount of \( f \) per unit produced.

(a) What is implied, if anything, about the slopes of the marginal revenue and marginal cost curves in this model?

(b) What refutable comparative statics implications are forthcoming, if any?

(c) Suppose this firm was not a monopolist, but rather, sold its total output in a competitive market at price \( p \). What differences would exist in the observable implications of the model in the competitive versus the monopolistic case?

(d) Suppose this competitive output price rose. Will the output in each plant increase?

(e) Returning now to the monopolistic case, suppose this monopolist decided to raise the price charged to consumers. What effect would this have on the output of each plant... hey, wait a minute... does this make any sense?

(f) Suppose the total revenue received by this (monopolistic) firm depends in some complicated way on outputs in both plants, rather than simply on the sum of those two outputs. What observable differences, if any, are implied by this change in assumptions?

(g) Suppose that output at the U.S. plant \( (y_1) \) is held fixed at the previously profit-maximizing level and the tax on Canadian output is increased. How does the resulting magnitude of the response in production at the Canadian plant compare with the response when U.S. output is unconstrained? (Again, assume the monopoly case.)

12. Prove, using Eq. (4-42), that the long-run supply curve of a competitive firm is more elastic than the supply curve in which one factor is held fixed at a previously profit-maximizing level.

13. Consider a profit-maximizing firm with production function \( y = f(x_1, x_2) \) that sells its output competitively at price \( p \). The firm obtains input \( x_1 \) at a competitively determined unit wage \( w_1 \), but the firm faces an upward-sloping supply function for \( x_2 \) given by \( w_2 = w_2^* + k x_2 \), where \( w_1, w_2^* \), \( p \), and \( k \) are positive parameters.

(a) Derive the first- and (sufficient) second-order conditions and explain the derivation of the explicit choice functions implied in this model. Characterize each of these
choice functions as a demand function, a supply function, or neither, and explain. Is the "law of diminishing marginal product" implied for each factor?
(b) Derive the comparative statics results available for the parameter $w_1$. What refutable implications are forthcoming, if any?
(c) How will the use of $x_2$ by this firm respond to an increase in $w_2$?
(d) Are the explicit choice functions homogeneous of some degree in some or all of the parameters? Prove that they either are or are not. What relation, if any, does homogeneity of factor demand or other similarly derived functions have, when it appears, to the homogeneity of the production function?
(e) Derive the comparative statics results for $w_2$ indicating which if any represent a refutable implication, and prove a "reciprocity" result involving the parameters $w_2$ and $w_1$.
(f) Suppose now that the firm is a monopolist in the output market, facing a demand curve $p = p(y)$, with total revenue $R(x_1, x_2) = p(y) f(x_1, x_2)$. What observable differences, if any, with regard to a firm's responses to changes in factor prices would exist between this monopolistic model and the previous model of profit maximization in a competitive output market?
(g) Returning to the competitive output market model, suppose $x_2$ is held fixed at its previous profit-maximizing level. Show how the "short-run" choice function for $x_1, x_1^*(w_1, p, w_2)$ is derived, and prove that it is downward-sloping in $w_1$.
(h) The supply function of this firm can be defined in the long and short run as $y^*(w_1, w_2, p, k)$ and $y^*(w_1, p, x_2^*)$, respectively. Show how these supply functions are derived and then explain clearly the identity

$$y^*(w_1, w_2, p, k) = y^*(w_1, p, x_1^*(w_1, w_2, p, k))$$

Use this result to show that the long-run supply function is more elastic than the short-run supply function.

14. Consider a profit-maximizing firm that employs one input $x$ and produces two outputs $y_1$ and $y_2$ according to the production frontier $f(y_1, y_2) = x$. It sells its outputs at prices $p_1$ and $p_2$, respectively, and purchases the input $x$ at price $w$. The firm obtains input $x$ at a competitively determined unit wage $w$ and sells output $y_2$ in a competitive market at price $p_2$. However, the firm faces a downward-sloping demand curve for $y_1$, given by

$$p_1 = p_1^0 - kx_1$$

where $p_1^0, p_2, w$, and $k$ are positive parameters.
(a) Derive the first and (sufficient) second-order conditions, and explain the derivation of the explicit choice functions implied in this model. Characterize each of these choice functions as a demand function, a supply function, or neither, and explain.
(b) Derive the comparative statics results available for the parameter $p_2$. What refutable implications are forthcoming, if any?
(c) Are the explicit choice functions homogeneous of some degree in some or all of the parameters? Explain. If so, derive a relationship of elasticities for these functions. What relation, if any, does homogeneity of the explicit choice functions have, when it appears, to the homogeneity of the production relationship?
(d) Derive the comparative statics relations for the parameter $p_1^0$ and interpret these results. What refutable implications, if any, appear?
(e) How will the use of $y_1$ by this firm respond to an increase in $p_1$?
(f) Derive a "reciprocity" result involving the parameters $p_1^0$ and $p_2$.
(g) Suppose now that the firm is a monopsonist in the input market; i.e., as it purchases more $x$, it bids up the wage $w$. Assume the firm faces a supply curve $w = w(x)$, with total cost $C(y_1, y_2) = w(f(y_1, y_2)) f(y_1, y_2)$. What differences with regard to
changes in output in response to a change in an output price would exist, if any, in the observable implications of such a model and the model of profit maximization in a competitive input market?

(h) Returning to the competitive input market model, suppose \( y_1 \) is held fixed at its previous profit-maximizing level, denoted \( y_1^* \). Show how the "short-run" choice function for \( y_2 \), \( y_2^* (p_2, w, y_1^*) \) is derived, and show that it is upward-sloping in \( p_2 \).

(i) Explain clearly the identity
\[
y_2^* (p_1^0, p_2, w, k) = y_2^* (p_2, w, y_1^* (p_1^0, p_2, w, k))
\]
Use this result to show that the long-run choice function for \( y_2 \) is more elastic than the short-run function.

4.7 ANALYSIS OF FINITE CHANGES: A DIGRESSION

The downward slope of the factor demand curves can be derived without the use of calculus, on the basis of simple algebra. Suppose that at some factor price vector \( (w_1^0, w_2^0) \), the input vector that maximizes profits is \( (x_1^0, x_2^0) \). This means that if some other input levels \( (x_1^0, x_2^0) \) were employed at the factor prices \( (w_1^0, w_2^0) \), profits would not be as high. Algebraically, then,

\[
pf (x_1^0, x_2^0) - w_1^0 x_1^0 - w_2^0 x_2^0 \geq pf (x_1^0, x_2^0) - w_1^0 x_1^0 - w_2^0 x_2^0
\]

However, there must be some factor price vector \( (w_1^0, w_2^0) \) at which the input levels \( (x_1^0, x_2^0) \) would be the profit-maximizing levels to employ. Since \( (x_1^0, x_2^0) \) leads to maximum profits at \( (w_1^0, w_2^0) \), any other level of inputs, in particular \( (x_1^0, x_2^0) \), will not do as well. Hence,

\[
pf (x_1^0, x_2^0) - w_1^0 x_1^0 - w_2^0 x_2^0 \geq pf (x_1^0, x_2^0) - w_1^0 x_1^0 - w_2^0 x_2^0
\]

If these two inequalities are added together, all the production function terms cancel, leaving (after multiplication through by \(-1\)):

\[
w_1^0 x_1^0 + w_2^0 x_2^0 + w_1^0 x_1^0 + w_2^0 x_2^0 \leq w_1^0 x_1^0 + w_2^0 x_2^0 + w_1^0 x_1^0 + w_2^0 x_2^0
\]

If the terms on the right-hand side are brought over to the left and the \( w_i \)'s factored, the result is

\[
w_1^0 (x_1^0 - x_1^0) + w_2^0 (x_2^0 - x_1^0) + w_1^0 (x_1^0 - x_1^0) + w_2^0 (x_2^0 - x_2^0) \leq 0
\]

However, this can be factored again, using the terms \( (x_1^0 - x_1^0) \), et cetera [note that \( x_1^0 - x_1^0 = -(x_1^0 - x_1^0) \)], yielding

\[
(w_1^0 - w_1^0) (x_1^0 - x_1^0) + (w_2^0 - w_2^0) (x_2^0 - x_2^0) \leq 0
\]

Suppose now that only one factor price, say \( w_1 \), changed. Then Eq. (4-43) becomes

\[
(w_1^0 - w_1^0) (x_1^0 - x_1^0) \leq 0
\]

or

\[
(\Delta w_1) (\Delta x_1) \leq 0
\]
Equation (4-44) says that the changes in factor utilization will move oppositely to changes in factor price; i.e., the law of demand applies to these factors. Note that if the profit maximization point is unique, the weak inequalities can be replaced with strict inequalities.

This is the type of algebra that underlies the theory of revealed preference, to be discussed later. Curiously enough, this analysis cannot be used to show the second law of demand, that (factor) demands will become more elastic as more factors are allowed to vary. As was stated in Sec. 4.4, that theorem was a strictly local phenomenon, holding only at a point. The previous analysis, which makes use of finite changes, turns out to be insufficiently powerful to analyze the Le Châtelier effects, i.e., the second law of demand.

APPENDIX

TAYLOR SERIES FOR FUNCTIONS OF SEVERAL VARIABLES

In Chap. 2, we indicated that it is sometimes possible to represent a function of one variable \( x \) by an infinite power series

\[
f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)(x - x_0)^2}{2!} + \ldots \tag{4A-1}
\]

It is, however, always possible to represent a function in a finite power series:

\[
f(x) = f(x_0) + f'(x_0)(x - x_0) + \ldots + \frac{f^{(n)}(x^*)(x - x_0)^n}{n!} \tag{4A-2}
\]

where \( x^* \) lies between \( x_0 \) and \( x \), that is; \( x^* = x_0 + \theta(x - x_0) \) where \( 0 \leq \theta \leq 1 \). These formulas were used to derive the necessary and sufficient conditions for a maximum (or minimum) at \( y = f(x) \).

Let us generalize these formulas to the case of, first, two independent variables; that is, \( y = f(x_1, x_2) \). This is accomplished by an artifice similar to the derivation of the maximum conditions in the text. Consider \( f(x_1, x_2) \) evaluated at some point \( x^0 = (x_1^0, x_2^0) \), that is, \( f(x_1^0, x_2^0) \). Let us now move to a new point, \( (x_1^0 + h_1, x_2^0 + h_2) \), where we can consider \( h_1 = \Delta x_1, h_2 = \Delta x_2 \). If we let

\[
y(t) = f(x_1^0 + h_1 t, x_2^0 + h_2 t) \tag{4A-3}
\]

then when \( t = 0, f(x_1, x_2) = f(x_1^0, x_2^0) \), and when \( t = 1, f(x_1, x_2) = f(x_1^0 + h_1, x_2^0 + h_2) \). If \( h_1 \) and \( h_2 \) take on arbitrary values, any point in the \( x_1, x_2 \) plane can be reached. We can therefore derive a Taylor series for \( f(x_1, x_2) \) by writing one for \( y(t) \), around the point \( t = 0 \). In terms of finite sums,

\[
y(t) = y(0) + y'(0)t + \frac{y''(0)t^2}{2!} + \ldots + \frac{y^{(m)}(0)t^m}{m!} \tag{4A-4}
\]
where \(0 \leq |t^*| \leq |t|\). Setting \(t = 1\), we have
\[
y(1) = f(x^0_1 + h_1, x^0_2 + h_2)
\]
\[
y(0) = f(x^0_1, x^0_2)
\]
\[
y'(0) = f_1(x^0_1, x^0_2)h_1 + f_2(x^0_1, x^0_2)h_2
\]
\[
y''(0) = \sum_{i=1}^{2} \sum_{j=1}^{2} f_{ij}(x^0_1, x^0_2)h_i h_j
\]
\[
\vdots
\]

Therefore, Eq. (4A-4) becomes
\[
f(x^0_1 + h_1, x^0_2 + h_2) = f(x^0_1, x^0_2) + \sum f_i h_i + \frac{\sum \sum f_{ij} h_i h_j}{2!} + \ldots
\]
\[
+ \frac{\sum \cdots \sum f_{ij} \cdots (x^*_i, x^*_j) h_i h_j \cdots}{m!} \tag{4A-5}
\]

where the last term is an \(m\)-sum of \(m\) partials times a product of the appropriate \(m\) \(h_i\)'s. The value of \(x = (x_1, x_2)\) at which the last term is evaluated is some \(x^*\) between \(x^0\) and \(x^*\), i.e., where
\[
x^*_i = x^0_i + \theta (x_i - x^*_i) \quad i = 1, 2 \tag{4A-6}
\]
with \(0 \leq \theta \leq 1\). Formula (4A-5) generalizes in an obvious fashion to functions of \(n\) variables. Then the sums run from 1 through \(n\) instead of merely from 1 to 2.

**Concavity and the Maximum Conditions**

**FIRST-ORDER NECESSARY CONDITIONS.** We can derive the first-order conditions for maximizing \(y = f(x_1, x_2)\) at \(x^0_1, x^0_2\) by considering (4A-5) with the last term being the linear term. In that case, we have the mean value theorem for \(f(x_1, x_2)\):
\[
f(x^0_1 + h_1, x^0_2 + h_2) - f(x^0_1, x^0_2) = f_1(x^*) h_1 + f_2(x^*) h_2 \tag{4A-7}
\]

If \(f(x_1, x_2)\) has a maximum at \(f(x^0_1, x^0_2)\), then the left-hand side of Eq. (4A-7) is necessarily nonpositive (negative for a unique maximum) for all \(h_1, h_2\) (not both 0). Letting \(h_2 = 0\) first, we see that
\[
f_1(x^*_1, x^*_2) \leq 0 \quad h_1 > 0
\]
and
\[
f_1(x^*_1, x^*_2) \geq 0 \quad h_1 < 0
\]
This can happen (if \( f_1 \) is continuous) only if \( f_1(x_0^0, x_2^0) = 0 \). Similarly, we deduce \( f_2 = 0 \). This procedure generalizes to the case of \( n \) variables in an obvious fashion.

**The Second-Order Conditions; Concavity.** If \( f(x_1, x_2) \) is a concave function at a stationary value, then \( f(x_1, x_2) \) has a maximum there. A concave function of two (or \( n \)) variables is defined as in Chap. 2 for one variable. A function \( f(x_1, x_2) \) is concave if it lies above (or on) the chord joining any two points.

If \( x^0 = (x_1^0, x_2^0) \) and \( x^1 = (x_1^1, x_2^1) \) are any two points in the \( x_1x_2 \) plane, 
\[
x' = tx^0 + (1 - t)x^1, \quad 0 \leq t \leq 1
\]
represents all points on the straight line joining \( x^0 \) and \( x^1 \). Algebraically, then, \( f(x_1, x_2) \) is concave if for any \( x^0, x^1 \),
\[
f(tx^0 + (1 - t)x^1) \geq tf(x^0) + (1 - t)f(x^1), \quad 0 \leq t \leq 1
\]
If the strict inequality holds (for \( 0 < t < 1 \)), implying no "flat" sections, the function is said to be strictly concave. Convex and strictly convex functions are defined analogously, with the direction of the inequality sign reversed. These definitions all generalize in an obvious way for functions of \( n \) variables; simply let \( x^0 \) and \( x^1 \) represent vectors in \( n \)-space.

For differentiable functions, concave functions lie below (or on) the tangent plane. Letting
\[
y(t) = f(x_1^0 + h_1t, x_2^0 + h_2t)
\]
as before, and recalling Eqs. (2.14) in Chap. 2, strict concavity implies
\[
y(t) - y(0) - y'(0)t < 0
\]
for all nontivial \( h_1, h_2 \). Applying (4A-8) with \( t = 1, x_i = x_i^0 + h_i, i = 1, 2 \),
\[
f(x_1, x_2) - f(x_1^0, x_2^0) - f_1(x_1^0, x_2^0)h_1 - f_2(x_1^0, x_2^0)h_2 < 0
\]
Taking the Taylor series expansion (4A-5) to the second-order term and rearranging slightly yields
\[
f(x_1, x_2) - f(x_1^0, x_2^0) - f_1(x_1^0, x_2^0)h_1 - f_2(x_1^0, x_2^0)h_2 = \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} f_{ij}(x_1^0, x_2^0)h_i h_j
\]
From (4A-9),
\[
\sum_{i=1}^{2} \sum_{j=1}^{2} f_{ij}(x_1^0, x_2^0)h_i h_j < 0
\]
for all \( h_i, h_j \) not both 0. Hence, strict concavity implies (4A-11). If the \( h_i \)’s are made smaller and smaller, \( f_{ij}(x_1^0, x_2^0) \) converges toward \( f_{ij}(x_1^0, x_2^0) \). We can deduce that concavity at \( x_1^0, x_2^0 \) implies that
\[
\sum_{i=1}^{2} \sum_{j=1}^{2} f_{ij}(x_1^0, x_2^0)h_i h_j \leq 0
\]
similarly, we deduce an obvious fashion.

\( \frac{x_1^0}{x_2} \) is a concave function. A concave function is defined by two points.

in the \( x_1x_2 \) plane, a straight line joining \( x_0^1 \),

\( i \leq 1 \)

sections, the function are defined. These definitions imply let \( x_0^0 \) and \( x_1^1 \)

(or on) the tangent

\[(4A-3)\]

implies

\[(4A-8)\]

\( h_i, i = 1, 2, \frac{h}{0} \)

and rearranging

\[\sum_{i=1}^{2} f_{ij}(x_1^*, x_2^*) h_i h_j \]

\[(4A-10)\]

\[(4A-11)\]

the \( h_i \)'s are made to can deduce that

\[(4A-12)\]

for all \( h_i, h_j \), but not that this expression is strictly negative at \( (x_1^0, x_2^0) \). If this double sum is strictly negative, then \( f(x_1, x_2) \) must be strictly concave. Similar remarks hold for convex functions.

If \( f(x_1, x_2) \) has an extremum at \( (x_1^0, x_2^0) \), then \( f_1 = f_2 = 0 \) there. Equation \((4A-10)\) then reveals how the second partials are related to a maximum or minimum position. Again, all the results of this section generalize to functions of \( n \) variables by simply having the sums in expressions \((4A-5), (4A-9), (4A-10), \) etc., run from 1 to \( n \), instead of just from 1 to 2.

SELECTED REFERENCES


