14.1 NONNEGATIVITY

In the previous pages we have largely ignored the issues raised by constraining the variables in a maximization model to be nonnegative. In the model of the firm, for example, we did not consider the possibility that simultaneous solution of the first-order equations might lead to negative values of one or more inputs. Such an occurrence would nullify the condition for profit maximization that wages be equal to marginal revenue product. In a more general sense, there are many factors of production that a firm chooses not to use at all. Similarly, consumers choose to consume only a small fraction of the myriad of consumer goods available. It is possible to characterize mathematically the conditions under which nonnegativity becomes a binding constraint. It might be remarked first, however, that since the refutable comparative statics theorems are concerned with how choice variables change when parameters change, the comparative statics of variables not chosen is fairly trivial. In a local sense (the evaluation of the partial derivatives of the choice functions at a given point) these variables continue not to be chosen; that is, $\frac{3 \pi^*}{3 \alpha_j} = 0$ for these variables. In a global sense, e.g., price changes of finite magnitude, factors or goods previously not chosen may enter the relevant choice set. For these situations, more powerful assumptions must be made to yield refutable theorems than in our previous discussions, where strictly local phenomena were analyzed.

Consider the monopolist of the first chapter. A profit function of the type

$$\pi(x) = R(x) - C(x)$$

is asserted to be maximized, where $R(x)$ and $C(x)$ denote, respectively, the revenue and cost associated with a given level of output $x$. (We are ignoring the tax aspect of the model, as it is not germane to this discussion.) The first-order conditions for a maximum of $\pi(x)$ are

$$\pi'(x) = R'(x) - C'(x) = 0$$

(14-2)

However, this condition is meant to apply only to those situations where the solution to (14-2) is nonnegative. The firm might choose to produce zero output, however, if, for example, $R'(x) < C'(x)$ for all $x > 0$. In that case, where the marginal revenue is less than marginal cost, increasing output reduces profits $\pi(x)$. The existence of a maximum of profits (not necessarily positive profits, another issue entirely) at some positive level of output $x^*$ presupposes that for some $0 \leq x \leq x^*$, MR > MC; that is, $R'(x) > C'(x)$ so that it "paid" for the firm to start operations in the first place. The only reason the profit maximum would occur at $x = 0$ is that MR(0) ≤ MC(0). That is, if maximum $\pi$ occurs at $x = 0$, then $x^* = R'(x) - C'(x) \leq 0$ at $x = 0$. The converse is not being asserted; it is in fact false. If $R'(x) - C'(x) < 0$ at $x = 0$, this does not imply that an interior maximum cannot occur at some $x$ distant from the origin. Again, the only aspect of the firm's behavior under consideration here is the attainment of maximum profits, not whether the firm shall exist or not [presumably dependent upon $\pi(x) > 0$).

Let us summarize this condition for maximization of functions of one variable. Consider some function $y = f(x)$. Then the first-order condition for $f(x)$ to achieve a maximum subject to the nonnegativity constraint $x \geq 0$ is

$$f'(x) \leq 0 \quad \text{if} \quad f'(x) < 0 \quad \text{then} \quad x = 0$$

(14-3)

Alternatively, one can express the same idea as

$$f'(x) \geq 0$$

$$x f'(x) \geq 0$$

(14-4a)

(14-4b)

Geometrically, the situation is as depicted in Fig. 14-1. In Fig. 14-12 the usual, interior maximum is illustrated. This solution is called an interior maximum because the value of $x$ that maximizes $f(x)$ does not lie on the boundary of the set over which $x$ is defined (here, the nonnegative real axis; its only boundary is the point $x = 0$). The set of positive real numbers is the interior of this domain of definition of $x$: hence the terminology. In Fig. 14-1b and c, corner solutions are depicted. That is, the maximum value of $f(x)$, for $x \geq 0$, occurs when $x = 0$. (The fact that the function in Fig. 14-1b achieves a regular maximum at a negative value of $x$ is irrelevant.) When the maximum occurs at $x = 0$, it is impossible to have $f'(x) > 0$ there. If $f'(0) > 0$, increasing $x$ would increase $f(x)$ and $f(0)$ could not be a maximum. However, it is possible that $f'(0) = 0$, as in Fig. 14-1c. There, the
to an equality constraint, ordinary Lagrangian methods can be used to derive the first-order conditions.

The constraint \( x \geq 0 \) is equivalent to

\[
    x - s^2 = 0
\]

(14-5)

where \( s \) takes on any real value. When \( s \neq 0 \), an interior solution is implied, since \( x = s^2 > 0 \). When \( s = 0 \), a corner solution is present.

We can now state this as the constrained maximum problem:

maximize

\[
    y = f(x)
\]

subject to

\[
    x - s^2 = 0
\]

The Lagrangian for this problem is

\[
    \mathcal{L} = f(x) + \lambda(x - s^2)
\]

(14-6)

Taking the first partials of \( \mathcal{L} \) with respect to \( x \), \( s \), and \( \lambda \) gives

\[
    \begin{align*}
    \mathcal{L}_x &= f'(x) + \lambda = 0 \\
    \mathcal{L}_s &= -2\lambda s = 0 \\
    \mathcal{L}_\lambda &= x - s^2 = 0
    \end{align*}
\]

(14-7)

From Eq. (14-7b) we see that if \( s \neq 0 \), that is, an interior solution is obtained, then \( \lambda = 0 \) and hence from (14-7a), \( f'(x) = 0 \). Thus, as expected, the usual condition \( f'(x) = 0 \) is obtained for noncorner solutions. Using the second-order conditions for constrained maximization, we can show that \( \lambda \geq 0 \). The second-order condition is that

\[
    \mathcal{L}_{xx}h_x^2 + 2\mathcal{L}_{xs}h_xh_s + \mathcal{L}_{ss}h_s^2 \leq 0
\]

(14-8)

for all \( h_x, h_s \) satisfying

\[
    g_xh_x + g_sh_s = 0
\]

(14-9)

where \( g(x, s) = x - s^2 \), the constraint. From the Lagrangian (14-6), \( \mathcal{L}_{xx} = f''(x) \), \( \mathcal{L}_{xs} = 0 \), \( \mathcal{L}_{ss} = -2\lambda \). From the constraint \( g(x, s) = x - s^2 \), \( g_x = 1 \), \( g_s = -2s \).

Hence, (14-8) and (14-9) become

\[
    f''(x)h_x^2 - 2\lambda h_s^2 \leq 0
\]

(14-10)

for all \( h_x, h_s \) satisfying

\[
    h_x - 2sh_s = 0
\]

(14-11)

We already know from Eq. (14-7b) that if \( s \neq 0 \), then \( \lambda = 0 \). Suppose now that \( s = 0 \). Then from Eq. (14-11), \( h_x = 0 \), but no restriction is placed on \( h_s \). When we

nonnegativity constraint is nonbinding. That is, the maximum \( f(x) \) would occur at \( x = 0 \) anyway, even without the restriction \( x \geq 0 \). Thus, if a maximum occurs when \( x > 0 \), \( f'(x) = 0 \). If the maximum occurs when \( x = 0 \), then necessarily \( f'(x) \leq 0 \). This condition is expressed in relation (14-3) or, equivalently, (14-4).

These more general first-order conditions can be derived algebraically by the device known as adding a slack variable. The constraint \( x \geq 0 \) is an elementary form of the more general inequality constraint \( g(x) \geq 0 \). By converting this inequality
use $h_x = 0, h_z = 0$, Eq. (14-10) becomes

$$-2\lambda h_x^2 \leq 0$$

implying, since $h_z^2 > 0$,

$$\lambda \geq 0$$

(14-12)

We now have a complete statement of the first-order conditions for maximizing $f(x)$ subject to $x \geq 0$. From (14-7a), since $\lambda \geq 0$,

$$f'(x) \leq 0$$

(14-13)

If $f'(x) < 0$, then $\lambda > 0$. From (14-7b) $s = 0$ and thus $x = 0$ from (14-7c). Therefore,

$$f'(x) < 0 \quad x = 0$$

(14-14)

Equations (14-13) and (14-14) are equivalent to

$$f'(x) \leq 0$$

(14-15)

$$xf'(x) = 0$$

(14-16)

commonly written

$$f'(x) \leq 0 \quad \text{if} \quad x = 0$$

Notice that if the maximum occurs at $x = 0$, no restrictions on $f''(0)$ are implied. In Fig. 14-1b, $f'(x)$ could be either convex (as drawn) or concave, and the maximum would still occur at $x = 0$.

These conditions can also be derived using the determinantal conditions on the bordered Hessian of second partials of $L$:

$$|L| = \begin{vmatrix}
L_{xx} & L_{xz} & L_{x}\nL_{xz} & L_{zz} & L_{z}\nL_x & L_z & 0
\end{vmatrix} > 0$$

Using the values previously calculated for these partials, we have

$$|L| = \begin{vmatrix}
f''(x) & 0 & 1 \\
0 & -2\lambda & -2s \\
1 & -2s & 0
\end{vmatrix} = -4s^2 f''(x) + 2\lambda \geq 0$$

(14-17)

From (14-17), if $s = 0$ (corner solution), $2\lambda \geq 0$; hence $\lambda \geq 0$. Thus, from (14-7a), $f'(x) \leq 0$.

The first-order conditions for obtaining a minimum value of $f(x)$ subject to $x \geq 0$ are obtained in a similar manner. One quickly shows that these conditions are

$$f'(x) \geq 0 \quad \text{if} \quad \lambda > 0, x = 0$$

(14-18)

That is, if a minimum occurs at $x = 0$, it must be the case that $f(x)$ is rising (or horizontal) at $x = 0$. Otherwise, i.e., if the function were falling at $x = 0$, making $x$ positive would lower the value of $f(x)$ and $f(x)$ could not have a minimum at $x = 0$.

Functions of Two or More Variables

The principles just delineated for maximization of functions of one variable generalize in an obvious manner to functions of two or more variables. Consider the problem

maximize

$$z = f(x_1, x_2)$$

subject to

$$x_1 \geq 0, \quad x_2 \geq 0$$

Let us now add slack variables $s_1^2, s_2^2$ in the manner of the first example. The problem then becomes one of maximization subject to two equality constraints:

maximize

$$y = f(x_1, x_2)$$

subject to

$$g^1(x_1, s_1) = x_1 - s_1^2 = 0$$
$$g^2(x_2, s_2) = x_2 - s_2^2 = 0$$

The Lagrangian for this problem is

$$L = f(x_1, x_2) + \lambda_1 (x_1 - s_1^2) + \lambda_2 (x_2 - s_2^2)$$

The first-order conditions for maximization are

$$L_{x_1} = f_1 + \lambda_1 = 0$$

(14-19a)
$$L_{x_2} = f_2 + \lambda_2 = 0$$

(14-19b)
$$L_{s_1} = -2\lambda s_1 = 0$$

(14-19c)
$$L_{s_2} = -2\lambda s_2 = 0$$

(14-19d)
$$L_{\lambda_1} = x_1 - s_1^2 = 0$$

(14-19e)
$$L_{\lambda_2} = x_2 - s_2^2 = 0$$

(14-19f)

From Eqs. (14-19c) and (14-19d), if either constraint is nonbinding, i.e., if $s_1 \neq 0$ or $s_2 \neq 0$, then, respectively, $\lambda_1 = 0, \lambda_2 = 0$. In that case ($x_1 > 0, x_2 > 0$), the ordinary first-order relations $f_1 = 0, f_2 = 0$ obtain.
We can show that \( \lambda_1 \geq 0, \lambda_2 \geq 0 \) by using the second-order conditions. For a constrained maximum,

\[
\sum_{i=1}^{2} \sum_{j=1}^{2} \ell \frac{\partial^2 \ell}{\partial x_i \partial x_j} h_i h_j + \sum_{i=1}^{2} \sum_{j=1}^{2} \ell \frac{\partial^2 \ell}{\partial x_i \partial x_j} k_i k_j \leq 0 \quad (14-20)
\]

for all values \( h_1, h_2, k_1, k_2 \) such that

\[
g_{h_1}^1 h_1 + g_{k_1}^1 k_1 = 0 \quad (14-21a)
\]

\[
g_{h_2}^2 h_2 + g_{k_2}^2 k_2 = 0 \quad (14-21b)
\]

By inspection of the Lagrangian (or Eqs. (14-19)) we have

\[
\frac{\partial \ell}{\partial x_i} = f_i \quad i, j = 1, 2
\]

\[
\frac{\partial \ell}{\partial k_i} = 0 \quad i, j = 1, 2
\]

\[
\ell_{x_{i,j}} = \begin{cases} 
-2\lambda_i & \text{if } i = j \\
0 & \text{if } i \neq j
\end{cases}
\]

\[
\ell_{x_{i,i}} = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{if } i \neq j
\end{cases}
\]

\[
\ell_{k_{i,j}} = \begin{cases} 
-2s_i & \text{if } i = j \\
0 & \text{if } i \neq j
\end{cases}
\]

Relations (14-20) and (14-21) therefore become

\[
\sum_{i=1}^{2} \sum_{j=1}^{2} f_i h_i h_j - 2\lambda_i k_i^2 - 2\lambda_j k_j^2 \leq 0 \quad (14-22)
\]

for all \( h_1, h_2, k_1, k_2 \) such that

\[
h_1 - 2s_1 k_1 = 0 \quad (14-23a)
\]

\[
h_2 - 2s_2 k_2 = 0 \quad (14-23b)
\]

We already know that if \( s_i \neq 0 \), then \( \lambda_i = 0 \). Suppose therefore that \( s_i = 0 \). Then from (14-23), \( h_i = 0 \). Then Eq. (14-22) becomes

\[
-2\lambda_i k_i^2 - 2\lambda_j k_j^2 \leq 0
\]

This must hold for all \( k_1, k_2 \). Setting \( k_1 = 0, k_2 = 0 \) in turn therefore yields

\[
\lambda_1 \geq 0 \quad (14-24a)
\]

\[
\lambda_2 \geq 0 \quad (14-24b)
\]

From the nonnegativity of the Lagrange multipliers, Eqs. (14-19c) and (14-19b) become

\[
f_1 \leq 0 \quad f_2 \leq 0
\]

And if \( f_i < 0 \) (meaning \( \lambda_i > 0 \)), then from (14-19c) and (14-19d), \( s_i = 0 \), and hence \( x_i = 0 \). Thus the first-order conditions for a maximum subject to nonnegativity constraints are

\[
f_i \leq 0 \quad \text{if } <, x_i = 0 \quad i = 1, 2 \quad (14-25)
\]

This reasoning generalizes to functions of \( n \) variables in a straightforward manner, yielding analogous results. The first-order conditions for maximize

\[
z = f(x_1, \ldots, x_n)
\]

subject to

\[
x_i \geq 0 \quad \text{some or all } i = 1, \ldots, n
\]

are

\[
f_i \leq 0 \quad \text{if } <, x_i = 0 \quad (14-26)
\]

for variables constrained to be nonnegative, and simply

\[
f_i = 0
\]

for variables not constrained to be nonnegative.

Let us see what these conditions imply for the profit-maximizing firm. We previously considered the model

\[
\pi = pf(x_1, x_2) - w_1 x_1 - w_2 x_2
\]

Let us now specify explicitly that the factors \( x_1 \) and \( x_2 \) can only be employed in positive amounts, as physical reality would dictate. With \( x_1, x_2 \geq 0 \), the first-order conditions for profit maximization become

\[
\pi_1 = pf_1 - w_1 \leq 0 \quad \text{if } <, x_1 = 0
\]

\[
\pi_2 = pf_2 - w_2 \leq 0 \quad \text{if } <, x_2 = 0 \quad (14-27)
\]

Equations (14-27) say that if the profit maximum occurs at zero input of some factor, then the value of the marginal product of that factor is less than its wage. This is in accord with intuition. If the marginal value product were initially greater than the

---

1 Some general mathematical treatments of the firm treat inputs as negative outputs. This type of black box approach to the theory of the firm generates a mathematical symmetry that is convenient in some analyses. Also, in more sophisticated models of the firm involving physical stocks of certain inputs, drawing down of some such stock (disinvestment) can be regarded as negative accumulation but probably still positive service flow from that stock.
wage of some factor, the firm could increase its profits by employing that factor in positive amounts.

Notice carefully the direction of implication intended by Eqs. (14-26) and, for the firm, (14-27). These relations do not say that if the marginal value product is initially, i.e., at \( x_i = 0 \), less than the wage of some factor, that factor will not be used. We might initially find \( p f_i < w_i \), but, as \( x_i \) increased, \( f_i \) might increase and then decrease, yielding \( p f_i = w_i \) at some finite, positive value of \( x_i \). The "law" of diminishing returns is in fact usually stated to allow this possibility; the usual assertion is that \( f_i \) declines after some level of use of \( x_i \) (holding the other factors constant). The preceding first-order equations say only that if the maximum of profits (or anything else) is observed to occur when \( p f_i < w_i \), then it must be the case that that input is not used; that is, \( x_i = 0 \). The converse of this statement is not implied by this analysis and will in general be false. These are strictly local conditions around the maximum position.

To illustrate this important point, consider a farmer who has to decide which of two tractors, a large model \( x_L \) or a small one \( x_s \), to purchase. Either one alone may yield positive profits, with a marginal value product initially greater than the rental wage. This particular farmer would never find it profitable to use two tractors. It turns out, say, that using only the smaller tractor yields the highest profits. At zero (or small) input levels of the other tractor, the marginal value product of either tractor is greater than the rental wage. But at maximum profits, \( x_j > 0 \), \( x_L = 0 \); at that point, \( p f s x_L < w_L \). But the nonuse of some factor does not imply that the value of the marginal product of that factor is always lower than its wage.

The generalized first-order conditions, while providing a conceptual generalization of the conditions for a maximum, are not useful for actually finding that maximum. As the previous paragraph indicates, these conditions describe the maximum position after the fact. They don't tell us in advance which variables will equal zero at the maximum position. Consider, for example, that firms usually employ only a few of the hundreds or thousands of potential factors of production available to them. Firms typically reject one type of machinery in favor of another, they set skill levels for employees, etc., rejecting certain factors outright. The preceding first-order conditions merely indicate that for the rejected factors, the marginal value product must have been lower than the wage, even at zero input levels. But that is precisely little to go on in predicting in advance exactly which factors will be employed and which factors will not.

More importantly, as indicated earlier, the only interesting refutable comparative-statics relations are those which predict a direction (or magnitude, if possible) of change in a choice variable as parameters change. The comparative statics of variables not chosen is rather elementary: \( \Delta x_j / \Delta a_j = 0 \) for all \( x_j \) not chosen, by definition. Hence the meaningful results that are forthcoming with mathematical model building will de facto be derived from the classical maximum conditions of first-order equalities. Models involving nonnegativity (or other inequality constraints) will in general require an algorithm for solution. That is, a given task involves error process will be required to see which, if any, constraints are in fact binding. In Chap. 17 on linear programming, an example of such an algorithm will be presented.

### 14.2 INEQUALITY CONSTRAINTS

Let us now consider the imposition of an inequality constraint \( g(x_1, x_2) \geq 0 \) in addition to the nonnegativity constraints in a two-variable problem. That is, consider maximize

\[
z = f(x_1, x_2)
\]

subject to

\[
g(x_1, x_2) \geq 0 \quad \text{and} \quad x_1 \geq 0, x_2 \geq 0
\]

(No loss of generality is involved by writing the constraint as \( g \geq 0 \); multiplying the constraint by \(-1\) reverses the sign.) Again, we first convert these inequalities to equalities, yielding the constrained maximum problem maximize

\[
z = f(x_1, x_2)
\]

subject to

\[
g(x_1, x_2) - s_1^2 = 0
\]

Here the slack variables are \( x_3, s_1, \) and \( s_2 \), the Lagrangian is

\[
L = f(x_1, x_2) + \lambda (g(x_1, x_2) - s_1^2) + \lambda_1 (x_1 - s_1^2) + \lambda_2 (x_2 - s_2^2)
\]

The first-order conditions for a maximum are thus

\[
\begin{align*}
L_{x_1} &= f_1 + \lambda_1 = 0 \\ L_{x_2} &= f_2 + \lambda_2 = 0 \\ L_{s_1} &= -2\lambda_1 x_1 = 0 \\ L_{s_2} &= -2\lambda_2 x_2 = 0
\end{align*}
\]

and the constraints

\[
\begin{align*}
\lambda_1 &= g(x_1, x_2) - x_3^2 = 0 \\ \lambda_{s_1} &= x_1 - s_1^2 = 0 \\ \lambda_{s_2} &= x_2 - s_2^2 = 0
\end{align*}
\]
Using exactly the same reasoning as before, we note from Eqs. (14-30) that if any constraint is nonbinding (holds as a strict inequality), then the associated Lagrange multiplier is 0. Suppose, at the maximum point, \( x_1, x_2 > 0 \), and \( g(x_1, x_2) > 0 \); then all these constraints turn out to be completely irrelevant. From Eqs. (14-30), \( \lambda = \lambda_1 = \lambda_2 = 0 \), and Eqs. (14-29) become the ordinary equations for unconstrained maximum, \( f_1 = f_2 = 0 \). If in fact \( g(x_1, x_2) = 0 \), that is, the constraint is binding, and \( x_1, x_2 > 0 \), then Eqs. (14-29) give the ordinary first-order conditions for a constrained maximum, \( \mathcal{L}_1 = f_1 + \lambda g_1 = 0 \), \( \mathcal{L}_2 = f_2 + \lambda g_2 = 0 \).

It must also be the case that \( \lambda, \lambda_1, \lambda_2 \geq 0 \). The second-order conditions for constrained maximum are

\[
\sum_{j=1}^{3} \sum_{i=1}^{3} \mathcal{L}_{x_i x_j} h_i h_j + 2 \sum_{j=1}^{3} \sum_{i=1}^{3} \mathcal{L}_{x_i x_j} h_i k_j + \sum_{j=1}^{2} \sum_{i=1}^{2} \mathcal{L}_{x_i x_j} k_i k_j \leq 0 \quad (14-32)
\]

for all \( h_1, h_2, h_3, k_1, k_2 \) satisfying

\[
\begin{align*}
g_1 h_1 + g_2 h_2 + g_3 h_3 &= 0 \quad (14-33a) \\
g_1^1 h_1 + g_2^1 k_1 &= 0 \quad (14-33b) \\
g_2^2 h_2 + g_3^2 k_2 &= 0 \quad (14-33c)
\end{align*}
\]

Now

\[
\mathcal{L}_{xx} = f_{ij} + \lambda g_{ij} = \mathcal{L}_{ij} \quad \mathcal{L}_{x_i x_j} = 0 \quad \text{for } i, j = 1, 2, 3
\]

and

\[
\mathcal{L}_{x_1 x_3} = -2 \lambda \quad \mathcal{L}_{x_3 x_3} = 0 \quad \text{if } i \neq 3
\]

Then the relations (14-32) and (14-33) become

\[
\sum_{j=1}^{2} \sum_{i=1}^{2} \mathcal{L}_{ij} h_i h_j - 2 \lambda h_2^2 - 2 \lambda \lambda_1 k_1^2 - 2 \lambda \lambda_2 k_2^2 \leq 0 \quad (14-34)
\]

for all \( h_1, h_2, h_3, k_1, k_2 \) such that

\[
\begin{align*}
g_1 h_1 + g_2 h_2 - 2 x_3 h_3 &= 0 \quad (14-35a) \\
h_1 - 2 \lambda_1 k_1 &= 0 \quad (14-35b) \\
h_2 - 2 \lambda_2 k_2 &= 0 \quad (14-35c)
\end{align*}
\]

Again, we already know that if \( s_1, s_2 \neq 0 \), then \( \lambda_1, \lambda_2 = 0 \), respectively. Also, if \( x_3 \neq 0 \), then \( \lambda = 0 \), from (14-30a). Therefore, suppose \( s_1 = s_2 = 0 \). Then, as before, from (14-35b) and (14-35c), \( \lambda_1 = h_2 = 0 \). Then (14-34) becomes

\[
-2 \lambda h_1^2 - 2 \lambda k_1^2 - 2 \lambda k_2^2 \leq 0
\]

Letting any two of \( h_3, k_1, k_2 = 0 \) [this is valid since (14-34) must hold for all \( h_i \)'s and \( k_i \)'s] yields

\[
\lambda \geq 0 \quad \lambda_1 \geq 0 \quad \lambda_2 \geq 0
\]

The first-order Eqs. (14-29) to (14-31) therefore can be stated as

\[
\begin{align*}
\mathcal{L}_i &= f_i + \lambda g_i \leq 0 \quad \text{if } i <, x_i = 0 \quad (14-36) \\
\mathcal{L}_i &= g_i (x_1, x_2) \geq 0 \quad \text{if } i >, \lambda = 0 \quad (14-37)
\end{align*}
\]

and we note that \( \lambda \geq 0 \).

These conditions generalize in a straightforward fashion to the case of \( n \) variables and \( m \) inequality constraints. In general, consider maximizing

\[
z = f(x_1, \ldots, x_n)
\]

subject to

\[
\begin{align*}
g^1(x_1, \ldots, x_n) &= 0 \\
g^m(x_1, \ldots, x_n) &= 0 \\
&\vdots \\
x_1, x_2, \ldots, x_n &\geq 0
\end{align*}
\]

There is no a priori need to restrict \( m \) to be less than \( n \) (as might be the case with equality constraints), since some (or all) of these constraints may turn out to be nonbinding.

Define the Lagrangian

\[
\mathcal{L} = f(x_1, \ldots, x_n) + \sum_{j=1}^{m} \lambda_j g^j(x_1, \ldots, x_n)
\]

Then the first-order conditions for a maximum are

\[
\begin{align*}
\mathcal{L}_i &= f_i + \sum_{j=1}^{m} \lambda_j g^j_i \leq 0 \quad \text{if } i <, x_i = 0 \quad (14-38) \\
\mathcal{L}_i &= g^j \geq 0 \quad \text{if } i >, \lambda_j = 0 \quad (14-39)
\end{align*}
\]

These relations are known as the Kuhn-Tucker conditions for a maximum subject to inequality constraints.\(^1\) Again, these conditions are not very useful for determining

the actual solution of such a problem. They are descriptions of the maximum position, after the fact, so to speak. If it turns out that at the maximum position \( f_i + \lambda g_i < 0 \), then \( x_i = 0 \). Nothing more is implied.

The conditions for a constrained minimum are similarly derived. Consider the problem

\[
\text{minimize} \quad z = f(x_1, \ldots, x_n)
\]

subject to

\[
g_j(x_1, \ldots, x_n) \leq 0 \quad j = 1, \ldots, m \\
x_i \geq 0 \quad i = 1, \ldots, n
\]

The constraints are written as \( \leq 0 \) to preserve symmetry. No loss of generality is involved; merely multiplying any constraint by \(-1\) reverses the sign of any constraint. Again, the Lagrangian, as before, is

\[
\mathcal{L} = f(x_1, \ldots, x_n) + \sum_{j=1}^{m} \lambda_j g_j(x_1, \ldots, x_n)
\]

The first-order conditions are then

\[
\mathcal{L}_{x_i} = f_i + \sum_{j=1}^{m} \lambda_j g_j \geq 0 \quad \text{if } x_i > 0, \quad i = 1, \ldots, n \\
\mathcal{L}_{\lambda_j} = g_j \geq 0 \quad \text{if } \lambda_j > 0, \quad j = 1, \ldots, m
\]

Writing the constraints as \( g_j \geq 0 \) ensures that \( \lambda_j \geq 0 \).

Let us illustrate these Kuhn-Tucker conditions using the model of a consumer who maximizes his or her utility \( U(x_1, x_2) \) subject to a budget constraint. Let us now assume that the consumer need not spend all of his or her money income. The model then becomes

\[
\text{maximize} \quad U(x_1, x_2)
\]

subject to

\[
p_1 x_1 + p_2 x_2 \leq M \quad x_1, x_2 \geq 0
\]

The Lagrangian for this problem is

\[
\mathcal{L} = U(x_1, x_2) + \lambda (M - p_1 x_1 - p_2 x_2)
\]

The constraint has been incorporated in the Lagrangian in the form \( M - p_1 x_1 - p_2 x_2 \geq 0 \), to conform with the previous analysis.

The first-order conditions are thus

\[
\mathcal{L}_1 = U_1 - \lambda p_1 \leq 0 \quad \text{if } <, x_1 = 0 \quad (14-40a) \\
\mathcal{L}_2 = U_2 - \lambda p_2 \leq 0 \quad \text{if } <, x_2 = 0 \quad (14-40b) \\
\mathcal{L}_\lambda = M - p_1 x_1 - p_2 x_2 \geq 0 \quad \text{if } >, \lambda = 0 \quad (14-40c)
\]

The Lagrange multiplier \( \lambda \) represents the consumer's marginal utility of money income. Briefly, suppose \( x_1, x_2 > 0 \). Then \( U_1 = \lambda p_1, U_2 = \lambda p_2 \), and

\[
\lambda = \frac{U_1}{p_1} = \frac{U_2}{p_2}
\]

The term \( U_1/p_1 \) represents the marginal utility, per dollar, of income spent on \( x_1 \). Likewise, \( U_2/p_2 \) represents the marginal utility of income spent on \( x_2 \). At a constrained maximum, these two ratios are equal, their common value being simply the marginal utility of money income.

Consider the last condition (14-40c). This can now be interpreted as saying that if the budget constraint is not binding, that is, \( p_1 x_1 + p_2 x_2 < M \) (the consumer doesn't exhaust his or her income), then \( \lambda \), the marginal utility of income, must be 0. The consumer is satiated in all commodities. This is confirmed by (14-40a) and (14-40b). If \( \lambda = 0 \), then \( U_1 = U_2 = 0 \); that is, the marginal utilities of both goods are 0. Hence, the consumer would not consume more of these goods even if they were given outright, i.e., free. This consumer is at a bliss point.

Now consider the situation where \( \lambda > 0 \) (the consumer would prefer to have more income) and \( x_1 = x_1^* > 0 \), but at the maximum point, \( U_1 - \lambda p_1 < 0 \) so that \( x_1 = x_1^* = 0 \). Assuming positive prices, we have at \( x_1^* > 0, \quad x_2^* > 0 \).

\[
\lambda = \frac{U_2}{p_2} > \frac{U_1}{p_1}
\]

Rearranging terms gives

\[
\frac{U_1}{p_1} < \frac{U_2}{p_2}
\]

This situation is depicted in Fig. 14-2. At any point, the consumer's subjective marginal evaluation of \( x_1 \), in terms of the \( x_2 \) the consumer would willingly forgo to consume an extra unit of \( x_1 \), is given by \( U_1/U_2 \), the ratio of marginal utilities. This is the (negative) slope of the indifference curve at any point. If the consumer chooses to consume no \( x_1 \) at all at the utility maximum, then the consumer's subjective marginal evaluation must be less than the value the market places on \( x_1 \). The market will exchange \( x_2 \) for \( x_1 \) at the ratio \( p_1/p_2 \). If, for example, \( p_1 = $6 \) and \( p_2 = $2 \), the market will exchange three units of \( x_2 \) for one unit of \( x_1 \). At zero \( x_1 \) consumption, a consumer valuing \( x_1 \) at only two units of \( x_2 \) would not be purchasing any \( x_1 \) at all at the utility maximum. In Fig. 14-2, this situation is represented by having the budget line cut the vertical \( x_2 \) axis at a steeper slope than the indifference curve.
For the Lagrangian
\[ \mathcal{L} = f(x_1, \ldots, x_n) + \sum_{j=1}^{m} \lambda_j g^j(x_1, \ldots, x_n) \]
the Kuhn-Tucker conditions are, again,
\[ \mathcal{L}_{x_i} = f_i + \sum_{j=1}^{m} \lambda_j g^j_i \leq 0 \quad \text{if } x_i = 0 \quad (14-38) \]
and
\[ \mathcal{L}_{\lambda_j} = g^j \geq 0 \quad \text{if } \lambda_j = 0 \quad (14-39) \]
Noting the direction of the inequalities, we see that these conditions are suggestive of the \textit{Lagrangian} function \( \mathcal{L}(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) \), achieving a \textit{maximum} in the \( x \) directions and a \textit{minimum} in the \( \lambda \) directions. That is, consider the Lagrangian above as just some function of \( x_i \)'s and \( \lambda_j \)'s. If \( \mathcal{L} \) achieved a \textit{maximum} with regard to the \( x_i \)'s, the first-order necessary conditions would be Eqs. (14-38). Likewise, if \( \mathcal{L} \) achieved a \textit{minimum} with respect to the \( \lambda_j \)'s, the first-order necessary conditions would be precisely Eqs. (14-39).

A point on a function which is a maximum in some directions and a minimum in the others is called a \textit{saddle point} of the function. The terminology is suggested by the shape of saddles: in the direction along the horse's backbone, the center of the saddle represents a minimum point, but going from one side of the horse to the other, the center of the saddle represents a maximum.

Consider a function \( f(x_1, \ldots, x_n, y_1, \ldots, y_m) \), or, more briefly, \( f(x, y) \), where \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_m) \). The point \((x^*, y^*)\) is said to be a saddle point of \( f(x, y) \) if
\[ f(x, y^*) \leq f(x^*, y^*) \leq f(x^*, y) \]

Let us now apply this concept to the Lagrangian above. If the Lagrangian \( \mathcal{L} = f(x_1, \ldots, x_n) + \sum_{j=1}^{m} \lambda_j g^j(x_1, \ldots, x_n) \) has a saddle point at some values \( x_i = x^*_i, i = 1, \ldots, n; \lambda_j = \lambda^*_j, j = 1, \ldots, m \) (briefly, at \( x = x^*, \lambda = \lambda^* \)), then, as a necessary consequence, the relations (14-38) and (14-39) are implied. That is, if
\[ \mathcal{L}(x, \lambda^*) \leq \mathcal{L}(x^*, \lambda^*) \leq \mathcal{L}(x^*, \lambda) \quad (14-41) \]
then it is being asserted that \( \mathcal{L}(x, \lambda) \) has a maximum in the \( x \) directions and a minimum in the \( \lambda \) directions. The first-order necessary conditions for such an extremum of \( \mathcal{L}(x, \lambda) \) are
\[ \mathcal{L}_{x_i} \leq 0 \quad \text{if } x_i = 0 \]
and
\[ \mathcal{L}_{\lambda_j} \geq 0 \quad \text{if } \lambda_j = 0 \]
However, the mere fact that two assertions [constrained maximum of \( f(x_1, \ldots, x_n) \) and saddle point of \( \mathcal{L}(x, \lambda) \)] imply the same conditions [Eq. (14.38) and (14.39)] does not imply that those two assertions are equivalent or that a particular one implies the other. It is the case, however, under fairly general mathematical conditions, that the saddle point criterion implies that \( f(x) \) has a constrained maximum. The converse is not true, however, unless stronger conditions are attached. If it assumed, in addition, that (1) \( f(x) \) and the \( g^j(x) \)'s are all concave functions and (2) there exists an \( x^0 > 0 \) such that \( g^j(x^0) = 0, \ j = 1, \ldots, m \) (this condition is known as Slater's constraint qualification), then if \( (x^*, \lambda^*) \) is a solution of the constrained maximum problem, \( (x^*, \lambda^*) \) is also a saddle point of the Lagrangian function.

This theorem is known as the Kuhn-Tucker saddle point theorem (there are actually many variants of it). Part of the proof appears in the Appendix to this chapter. Vector notation will be used throughout.

Suppose \( (x^*, \lambda^*) \) is in fact a saddle point of \( \mathcal{L}(x, \lambda) \). Then, by definition, for \( x \geq 0, \lambda \geq 0 \),

\[
f(x) + \lambda^* g(x) \leq f(x^*) + \lambda^* g(x^*)
\]

and

\[
f(x^*) + \lambda^* g(x^*) \leq f(x^*) + \lambda g(x^*)
\]

where \( g(x) \) means \( \sum_{j=1}^{m} \lambda_j g^j(x) \), the inner product of the vectors \( \lambda = (\lambda_1, \ldots, \lambda_m) \) and \( g(x) = (g^1(x), \ldots, g^m(x)) \). From (14.43), after canceling \( f(x^*) \) from both sides and rearranging, we have

\[
(\lambda - \lambda^*) g(x^*) \geq 0
\]

(14.44)

Since (14.44) must hold for any \( \lambda \), by hypothesis, for sufficiently large \( \lambda, \lambda - \lambda^* \geq 0 \) and hence

\[
g(x^*) \geq 0
\]

(14.45)

Thus we have shown that \( x^* \) is feasible; i.e., it satisfies the constraints of the maximum problem. Moreover, we can set \( \lambda = 0 \) in (14.44) (again, since this must hold for all \( \lambda \), obtaining, after multiplying by -1,

\[
\lambda^* g(x^*) \leq 0
\]

(14.46)

However, \( \lambda^* \geq 0, g(x^*) \geq 0 \). Therefore, in order to satisfy (14.46), it must be that

\[
\lambda^* g(x^*) = 0
\]

(14.47)

Now consider the first inequality, (14.42), which refers to the maximum in the \( x \) directions. When we use Eq. (14.47), (14.42) becomes

\[
f(x^*) \geq f(x) + \lambda^* g(x)
\]

(14.48)

However, \( \lambda^* \geq 0, \lambda \) and for any feasible \( x \), that is, an \( x \) which satisfies the constraints, \( g(x) \geq 0 \). Therefore, \( \lambda^* g(x) \geq 0 \), and thus

\[
f(x^*) \geq f(x)
\]

(14.49)

for any feasible \( x \). Therefore, \( x^* \) maximizes \( f(x) \) subject to the constraints \( g(x) \geq 0 \). We have therefore shown that the saddle point condition implies that a constrained maximum exists.

To repeat, the converse of the preceding is in general false. If conditions 1 and 2 above are added, viz., that \( f(x) \) and \( g^j(x), \ j = 1, \ldots, m \), are all concave and that there exists an \( x^0 \) such that \( g^j(x^0) > 0, \ j = 1, \ldots, m \), then the "converse" follows. The proof of this proposition unfortunately requires more advanced methods of linear algebra dealing with convex sets. It is presented in the Appendix to the chapter. Note, however, that the right-hand part of the saddle point inequality follows readily from the assumption of a constrained maximum. If \( x^*, \lambda^* \) are the values that maximize \( f(x) \) subject to \( g(x) \geq 0 \), then

\[
\mathcal{L}(x^*, \lambda^*) = f(x^*) + \lambda^* g(x^*)
\]

However, from the first-order conditions, \( \lambda^* g(x^*) = 0 \). Hence,

\[
\mathcal{L}(x^*, \lambda^*) = f(x^*)
\]

By definition

\[
\mathcal{L}(x^*, \lambda) = f(x^*) + \lambda g(x^*)
\]

But \( g(x^*) \geq 0, \lambda \geq 0 \) by assumption; thus

\[
\mathcal{L}(x^*, \lambda^*) = f(x^*) \leq f(x^*) + \lambda g(x^*) = \mathcal{L}(x^*, \lambda)
\]

i.e., the right-hand part of the relation (14.41).

Example. We shall show by example that achieving a constrained maximum does not imply that the Lagrangian has a saddle point there. Consider a consumer who maximizes the utility function \( U = x_1 x_2 \) subject to the constraint \( p_1 x_1 + p_2 x_2 = M \). Since the level (indifference) curves of the utility function, \( x_1 x_2 = U \), never cross the axes and \( U_1, U_2 > 0 \) for all positive \( x \), the consumer will in fact spend his or her entire income; that is \( M = p_1 x_1 + p_2 x_2 = 0 \). Thus, the problem is solved by formulating

\[
\mathcal{L} = x_1 x_2 + \lambda (M - p_1 x_1 - p_2 x_2)
\]

(14.50)

with first-order equations

\[
\begin{align*}
L_1 &= x_2 - \lambda p_1 = 0 \\
L_2 &= x_1 - \lambda p_2 = 0 \\
L_\lambda &= M - p_1 x_1 - p_2 x_2 = 0
\end{align*}
\]

(14.51)

The consumer's demand functions are found by first eliminating \( \lambda \):

\[
x_2 = \lambda p_1, \quad x_1 = \lambda p_2
\]
and thus
\[ \frac{x_2}{x_1} = \frac{p_1}{p_2} \]
or
\[ p_1 x_1 = p_2 x_2 \]
Substituting this relation into the budget constraint \( (P = 0) \) gives
\[ p_1 x_1 + p_1 x_1 = M \]
and thus
\[ x_1^* = \frac{M}{2 p_1} \]  
(14-52a)

Similarly,
\[ x_2^* = \frac{M}{2 p_2} \]  
(14-52b)

Also,
\[ \lambda^* = \frac{x_2^*}{p_1} = \frac{x_1^*}{p_2} = \frac{M}{2 p_1 p_2} \]  
(14-53)

We therefore find
\[ \mathcal{L}(x^*, \lambda^*) = x_1^* x_2^* + \lambda^* (M - p_1 x_1^* - p_2 x_2^*) \]  
(14-54)

However, the budget constraint is satisfied by \( x_1^*, x_2^* \), and thus
\[ \mathcal{L}(x^*, \lambda) = \frac{M}{2 p_1} \frac{M}{2 p_2} = \frac{M^2}{4 p_1 p_2} = U(x_1^*, x_2^*) \]

By definition
\[ \mathcal{L}(x^*, \lambda) = U(x_1^*, x_2^*) + \lambda (M - p_1 x_1^* - p_2 x_2^*) \]

Since the budget constraint is binding at \( x_1^*, x_2^* \),
\[ \mathcal{L}(x^*, \lambda) = U(x_1^*, x_2^*) = \frac{M^2}{4 p_1 p_2} = \mathcal{L}(x^*, \lambda^*) \]

Hence, the right-hand side of the saddle point is satisfied as an equality,
\[ \mathcal{L}(x^*, \lambda^*) = \mathcal{L}(x^*, \lambda) \]  
(14-55)

The left-hand side of the saddle point condition is not satisfied, however:
\[ \mathcal{L}(x, \lambda^*) = U(x_1, x_2) + \lambda^* (M - p_1 x_1 - p_2 x_2) \]
\[ = x_1 x_2 + \frac{M}{2 p_1 p_2} (M - p_1 x_1 - p_2 x_2) \]

If we let \( x_1 = x_2 = 0 \),
\[ \mathcal{L}(x, \lambda^*) = \frac{M^2}{2 p_1 p_2} > \frac{M^2}{4 p_1 p_2} = \mathcal{L}(x^*, \lambda^*) \]

The saddle point condition is violated because although \( U = x_1 x_2 \) is quasi-concave in \( x_1 \) and \( x_2 \), it is not concave. Thus the mere attainment of a constrained maximum is not sufficient for the Lagrangian to possess a saddle point at the maximum position.

14.4 NONLINEAR PROGRAMMING

The general class of problems involving maximization of a function subject to inequality and nonnegativity constraints is called non-linear programming problems. These problems, of the form

maximize
\[ y = f(x_1, \ldots, x_n) \]
subject to
\[ g^1(x_1, \ldots, x_n) \geq 0 \]
\[ \vdots \]
\[ g^m(x_1, \ldots, x_n) \geq 0 \]
\[ x_1, \ldots, x_n \geq 0 \]

do not contain specific enough structure to permit description of the solution. The determination of exactly which constraints will be binding and which will not makes this class of problems significantly more complex than the classical problem of maximizing a function subject to equality constraints with nonnegativity not imposed. Once it is shown which constraints are binding, the preceding problem reduces to a classical maximization problem, solvable (in principle—the equations may admit of no easily expressible solution) by standard Lagrangian techniques.

Solutions to nonlinear programming problems will be found only by some iterative procedure, i.e., an algorithm which leads one toward the maximum in a stepwise fashion. In general, such algorithms begin with an arbitrary feasible point, i.e., an \( x = (x_1, \ldots, x_n) \) which satisfies all the constraints including nonnegativity. Then in the neighborhood of that point some evaluation is made of how \( f(x) \) could be increased, e.g., by decreasing some \( x_i \)’s and increasing others. When a new point is reached, the evaluation is repeated. A successful algorithm is one which leads to the maximum position in a finite (but not astronomically large) number of steps.

A number of algorithms have been developed, assuming various specific structures on the \( f \) and \( g^i \) functions. The most famous is the simplex algorithm, developed by George Dantzig in 1947 for solving the class of linear programming problems.†

This type of problem results when \( f \) and the \( g_i \)'s are all linear functions, or maximize

\[
y = \sum_{i=1}^{n} p_i x_i
\]

subject to

\[
\sum_{j=1}^{m} a_{ij} x_j \leq b_i \quad i = 1, \ldots, m
\]

\[
x_j \geq 0 \quad j = 1, \ldots, n
\]

This class of problems will be investigated in Chap. 17 on linear general equilibrium models.

No general algorithm for all nonlinear programming problems exists. The specific algorithms that exist for some nonlinear problems are not of central interest to most economists and are outside the scope of this book. We shall only briefly indicate some structures for which algorithms have been more successful.

One of the central problems encountered in nonlinear programming problems is the determination of whether a local solution is in fact the global solution of the problem. That is, suppose \( f(x^*) \geq f(x) \) for all \( x \) in some neighborhood of \( x^* \). Then \( x^* \) is a local maximum. How can we be sure that \( x^* \) is the global solution, that is, \( f(x^*) \geq f(x) \), for all feasible \( x \)? In general, of course, one can't be sure, but under certain structures local solutions are in fact global solutions. Let us explore these circumstances.

Consider Fig. 14-3, in which a consumer attempts to maximize some utility function \( U(x_1, x_2) \) whose indifference curves \( U^1 \) and \( U^2 \) are shown. Suppose, contrary to the usual assumptions, that the budget constraint is not the usual linear form, \( p_1 x_1 + p_2 x_2 \leq M \), but the area bounded by the curved line \( MM' \). Given this situation, two local constrained maxima exist: \( x^* \) and \( x^{**} \). At \( x^* \), \( U(x^*) > U(x) \) for all \( x \) in some neighborhoods of \( x^* \). An iterative procedure which led to  \( x^* \) as the solution

to this problem might be insufficiently powerful to indicate that if the neighborhood is made large enough, some \( x \)'s will be found for which \( U(x) > U(x^*) \). In the given example, \( x^{**} \) is the global maximum, since clearly \( U(x^{**}) > U(x) \) for all other \( x \) in the budget set.

The problem of nonglobal maxima occurs here because points connecting \( x^* \) and \( x^{**} \) lie outside the feasible region, i.e., the set of all feasible \( x \)'s. That is, a straight line joining \( x^* \) and \( x^{**} \) contains points not admissible under the conditions of the model. A very important construct in analyses of nonlinear programming problems is therefore that of a convex set.

**Definition.** A set \( S \) is said to be convex if, for all \( x^1 \in S, x^2 \in S \) (the symbol "\( \in \)" means "belonging to"), the points \( x = kx^1 + (1 - k)x^2 \) belong to \( S \), for all \( 0 \leq k \leq 1 \).

Geometrically, a convex set is one such that all points along a straight line joining any two points in the set also belong to the set. The straight line joining any two points in the set never leaves the set. All squares, triangles, circles, spheres, and parallelograms are convex sets; sets like that depicted in Fig. 14-3, the points bounded by the axes and the curve \( MM' \), are nonconvex. The principal result on local versus global maxima is, as indicated in the above discussion, the following theorem.

**Theorem.** Let \( f(x), x = (x_1, \ldots, x_n) \) be a quasi-concave function defined over some convex set \( S \). Then if \( f(x^*) \) is a unique local maximum in \( S \), it is in fact the global maximum.

**Proof.** Suppose there exists an \( x^{**} \) such that \( f(x^{**}) > f(x^*) \). Then, by quasi-concavity,

\[
f(kx^* + (1 - k)x^{**}) \geq f(x^*) \quad 0 \leq k \leq 1
\]

By choosing \( k \) arbitrarily close to 1, the point \((kx^* + (1 - k)x^{**})\) becomes arbitrarily close to \( x^* \); yet, the function has a value there greater than or equal to its value at \( x^* \), a unique local maximum. This contradiction demonstrates the result.

Heuristically, if \( x^* \) and \( x^{**} \) are any two finitely separated points of local maxima, the chord joining them must lie in the convex set \( S \). The function evaluated along that chord must be at least as large as the smaller of \( f(x^*) \) and \( f(x^{**}) \). If, say, \( f(x^*) > f(x^{**}) \), points \( x \) arbitrarily near \( x^{**} \) must also yield \( f(x) > f(x^{**}) \), contradicting the assumption that \( x^{**} \) is a local maximum. If \( f(x^*) = f(x^{**}) \), the function must be constant along the chord joining \( x^* \) and \( x^{**} \). It follows, therefore, that if a local maximum is unique, it is the global maximum over the convex set \( S \).

Under what conditions will the set of variables over which a maximum problem is posed be convex? That is, under what conditions is the feasible region of a nonlinear programming problem a convex set? It is easy to show that if the constraints are all concave functions, the feasible region is in fact convex.

Consider the set \( S \) defined as the \( x = (x_1, \ldots, x_n) \) such that \( g(x) \geq a \), where \( a \) is any real number. Then \( S \) is convex; for consider any \( x_1, x_2 \) for which \( g(x_1) \geq a, g(x_2) \geq a \). From concavity

\[
g(kx_1 + (1 - k)x_2) \geq kg(x_1) + (1 - k)g(x_2) \geq ka + (1 - k)a = a \quad 0 \leq k \leq 1
\]
Therefore the point \( kx^1 + (1-k)x^2 \) lies in the set, and \( S \) is convex. If some functions \( g^1(x), \ldots, g^m(x) \) are all concave, then the set of \( x \)'s that satisfy

\[
g^1(x) \geq a_1
\]

\[
\vdots
\]

\[
g^m(x) \geq a_m
\]
simultaneously clearly also constitutes a convex set, as would be the case if nonnegativity constraints are added. (The intersection of convex sets is a convex set.) Hence, if the constraints of a programming problem are all concave, the feasible region will be a convex set. If the objective function is also concave, we can be assured that any local maximum is the global maximum of the model.

The principal application of the above theorem, to be discussed in the next chapter, is in the theory of linear programming in which \( f(x), g^1(x), \ldots, g^m(x) \) are all linear functions. In that case, the feasible region is convex, and an efficient algorithm for finding the solution to the problem has been developed.

14.5 AN "ADDING-UP" THEOREM

Many economic models have the general structure

maximize

\[ y = f(x_1, \ldots, x_n) \]

subject to

\[ g^1(x_1, \ldots, x_n) \leq b_1 \]

\[ \vdots \]

\[ g^m(x_1, \ldots, x_n) \leq b_m \quad x_1, \ldots, x_n \geq 0 \]

Let us now assume that \( f, g^1, \ldots, g^m \) are all homogeneous of the same degree \( r \). Assume that the problem admits of a solution found by standard Lagrange-Kuhn-Tucker techniques. The Lagrangian is

\[ L = f(x_1, \ldots, x_n) + \sum_{j=1}^{m} \lambda^*_j (b_j - g^j(x_1, \ldots, x_n)) \]  

(14-59)

The first-order conditions are therefore

\[ f_i \leq \sum_{j=1}^{m} \lambda^*_j g^j_i \quad \text{if} \quad <, \quad x_j = 0 \]

(14-57)

and

\[ b_j - g^j \geq 0 \quad \text{if} \quad >, \quad \lambda^*_j = 0 \]

(14-58)

Alternatively,

\[ f_i x^*_i = \sum_{j=1}^{m} \lambda^*_j g^j_i x^*_i \]

(14-59)

and

\[ b_j = g^j \]

(14-60)

Let us now sum (14-59) over \( i \) and (14-60) over \( j \). This yields

\[ \sum_{i=1}^{n} f_i x^*_i = \sum_{j=1}^{m} \lambda^*_j \sum_{i=1}^{n} g^j_i x^*_i \]

(14-61)

and

\[ \sum_{j=1}^{m} b_j \lambda^*_j = \sum_{j=1}^{m} g^j \lambda^*_j \]

(14-62)

Now let us use Euler's theorem. Since \( f \) and \( g^1, \ldots, g^m \) are all homogeneous of degree \( r \), \( \sum f_i x_i = rf_i, \sum g^j x_i = rg^j_i \), and hence from (14-61), letting \( y^* = f(x^*) \), we have

\[ ry^* = rf(x^*) = \sum_{j=1}^{m} \lambda^*_j r g^j_i (x^*) = r \sum_{j=1}^{m} \lambda^*_j b_j \]

or

\[ y^* = \sum_{j=1}^{m} \lambda^*_j b_j \]

(14-63)

Now from general envelope considerations,

\[ \lambda^*_j = \frac{\partial y^*}{\partial b_j} \]

If the constraint \( g^j(x) \leq b_j \) is thought of as a resource constraint, where \( b_j \) represents the amount of some resource used by the economy, \( \lambda^*_j = \partial y^*/\partial b_j \) represents the imputed rent, or shadow price, of that resource, measured in terms of \( y \). In other words, \( \lambda^*_j b_j \) can be thought of as the total factor cost of some factor associated with some resource allocation. Equation (14-63) then says that under these assumptions, the output being maximized can be allocated to each resource, with nothing left over on either side. This type of adding-up, or exhaustion-of-the-product, theorem appeared in the chapters on production and cost, when linear homogeneous production functions were involved. The preceding is a generalization of those results.

Moreover, consider the indirect objective function

\[ \phi(b_1, \ldots, b_m) = f(x^*_1, \ldots, x^*_n) = y^* \]
Since \( y^* = \sum_{j=1}^m \lambda_j^* b_j \) and \( \lambda_j^* = \frac{\partial y^*}{\partial b_j} = \frac{\partial \phi}{\partial b_j} \),

\[
\phi(b_1, \ldots, b_m) \equiv \sum_{j=1}^m \frac{\partial \phi}{\partial b_j} b_j
\]  

(14-64)

Therefore, under these conditions, the indirect objective function is homogeneous of degree 1 in the parameters \( b_1, \ldots, b_m \), from the converse of Euler's theorem.

PROBLEMS

1. Explain the error in the following statement: For a profit-maximizing firm, if the value of the marginal product of some factor is initially less than its wage, the factor will not be used. State the condition correctly.

2. Consider the constrained minimum problem

minimize

\[ z = f(x_1, x_2) \]

subject to

\[ g(x_1, x_2) \leq 0 \quad x_1, x_2 \geq 0 \]

Derive the Kuhn-Tucker first-order conditions for a minimum.

3. Consider the cost minimization problem

minimize

\[ C = w_1 x_1 + w_2 x_2 \]

subject to

\[ f(x_1, x_2) \geq y \quad x_1, x_2 \geq 0 \]

Derive and interpret the first-order conditions for a minimum. Under what conditions on the production function will the Lagrangian have a saddle point at the cost-minimizing solution?

4. Consider a consumer who maximizes the utility function \( U = x_1 x_2 \) subject to a budget constraint. Characterize the implied demand levels via the Kuhn-Tucker conditions, i.e., indicate when positive demand levels are present for both commodities, etc.

5. Consider the quadratic utility function \( U = a x_1^2 + 2 b x_1 x_2 + c x_2^2 \). Discuss the nature of the implied consumer choices for this utility function in terms of the values \( a, b, \) and \( c \).

6. Find the solution to the following nonlinear programming problem:

maximize

\[ y = x_1 x_2 \]

subject to

\[ x_1 + x_2 \leq 10 \quad x_1 + 2 x_2 \leq 18 \quad x_1, x_2 \geq 0 \]

7. Consider the nonlinear programming problem

maximize

\[ y = x_1 x_2 \]

subject to

\[ x_1 + x_2 \leq 10 \quad x_2 \leq k \quad x_1, x_2 \geq 0 \]

What is the maximum value of \( k \) for which that constraint is binding?

8. Solve

minimize

\[ y = x_1 + 2 x_2 \]

subject to

\[ x_1, x_2 \geq 8 \quad x_1 \geq 5 \quad x_1, x_2 \geq 0 \]

9. Solve Prob. 8 with \( x_1 \leq 5 \) replacing \( x_1 \geq 5 \).

10. An individual has the utility function \( U = x_1 x_2 \) for consumption in two time periods, with \( x_1 \) present consumption, \( x_2 \) next year's consumption. This person has an initial stock of capital of \$10, which can yield consumption along an "investment opportunities frontier," given by \( 2 x_2^2 + x_1^2 = 200 \). The person can, however, borrow and lend at some market rate of interest \( r \) to rearrange consumption.

(a) Explain why maximization of utility requires a prior maximization of wealth \( W \), where \( W = x_1 + x_2 / (1 + r) \). That is, explain why if \( W \) is not maximized, \( U(x_1, x_2) \) cannot be maximized.

(b) Suppose the consumer can borrow or lend at \( r = 30 \) percent. Find the utility-maximizing consumption choices. Is the consumer a borrower or a lender?

(c) Suppose the consumer can lend money at only 20 percent interest and can borrow at no less than 40 percent interest. What consumption plan maximizes utility, and what is the present value of that consumption?

APPENDIX

The proof that if \( f(x), g^1(x), \ldots, g^m(x) \) are all concave functions, then

\[
\mathcal{L}(x, \lambda^*) \leq \mathcal{L}(x^*, \lambda^*) = f(x^*)
\]

where \( x^* \) solves the maximum problem, is based on a famous theorem of convex set analysis. Consider two nonintersecting (disjoint) convex sets \( S_1 \) and \( S_2 \). It is geometrically obvious, though messy to prove, that a hyperplane (the generalization of a line in two dimensions, plane in three dimensions, etc.) can be passed between \( S_1 \) and \( S_2 \). This proposition is known as the separating hyperplane theorem. The theorem also holds if the sets are tangent at one point.

Consider Fig. 14-4; \( S_1 \) and \( S_2 \) are two convex sets that do not intersect. It is therefore possible to pass between them a line \( p_1 x_1 + p_2 x_2 = k \) or, in vector notation, \( px = k \). Figure 14-5 shows why such may not be the case if the sets are nonconvex.
there exist scalars \( p_1, \ldots, p_n \), not all zero, such that

\[
\sum_{i=1}^{n} p_i x_i^1 \leq \sum_{i=1}^{n} p_i x_i^2
\]  

(14A-1)

Let us return now to the saddle point problem. We are assuming that \( x^* \) maximizes \( f(x) \) subject to \( g^j(x) \geq 0, j = 1, \ldots, m, x \geq 0 \). We shall also assume Slater's constraint qualification that there exists an \( x^0 \geq 0 \) such that \( g^j(x^0) > 0, j = 1, \ldots, m \).

For any given \( x \), there exist the \( m + 1 \) values \( f(x), g^1(x), \ldots, g^m(x) \), an \((m + 1)\)-dimensional vector.

1. Define the set \( S_1 \) as the vectors \( U = (U_0, U_1, \ldots, U_m) \) such that \( U_0 \leq f(x), U_j \leq g^j(x), j = 1, \ldots, m, \) for all feasible \( x \).

2. Define \( S_2 \) as the vectors \( V = (V_0, V_1, \ldots, V_m) \) such that \( V_0 > f(x^*), V_j > 0, j = 1, \ldots, m. \)

The sets \( S_1 \) and \( S_2 \) are convex, disjoint sets. \( S_1 \) is convex because \( f, g^1, \ldots, g^m \) are all concave functions. The results at the end of Sec. 14.4 imply convexity for \( S_1 \); \( S_2 \) is convex because \( S_2 \) is essentially the positive quadrant in \( m + 1 \) space, except that the first coordinate, \( V_0 \), starts at \( f(x^*) \). Finally, since \( f(x^*) \geq f(x) \) and since \( V_0 > f(x^*) \), there can be no \( V \) vector that lies in \( S_1 \). The first coordinate, \( V_0 \), violates the definition of \( S_1 \).

Since \( S_1 \) and \( S_2 \) are disjoint convex sets, by the separating hyperplane theorem there exist scalars \( \lambda_0, \lambda_1, \ldots, \lambda_m \) such that

\[
\sum_{j=0}^{m} U_j \lambda_j \leq \sum_{j=0}^{m} V_j \lambda_j
\]  

(14A-2)

for all \( U \in S_1, V \in S_2 \). Moreover, although the point \((f(x^*), 0, \ldots, 0)\) is not in \( S_2 \), it is on the boundary of \( S_1 \), and hence the theorem applies to that point as well. The point \((f(x), g^1(x), \ldots, g^m(x))\) is in \( S_1 \). Hence, applying Eq. (14A-2) gives

\[
\lambda_0 f(x) + \sum_{j=1}^{m} \lambda_j g^j(x) \leq \lambda_0 f(x^*)
\]  

(14A-3)

It can be seen from Eq. (14A-2) that \( \lambda_0, \lambda_1, \ldots, \lambda_m \) are all nonnegative. The vectors \( U \) include the entire negative “quadrant,” or orthant, of this \( m + 1 \) space. Any of the \( U_j, j \) can be made arbitrarily large, negatively. Note that \( V_1, \ldots, V_m \) are all greater than 0. If any \( \lambda_j, j = 1, \ldots, m \), were negative, making that \( U_j \) sufficiently negative would violate the inequality (14A-2). Last, since \( f(x^*) \geq f(x) \) and since \( x^* \) maximizes \( f(x), \lambda_0 \geq 0 \) for essentially the same reasons.

Therefore, all the \( \lambda_j \)s in (14A-3) are nonnegative. Moreover, given the constraint qualification, \( \lambda_0 > 0 \); for suppose \( \lambda_0 = 0 \); then (14A-3) says that

\[
\sum_{j=1}^{m} \lambda_j g^j(x) \leq 0
\]
However, since the separating hyperplane theorem says that not all the $\lambda_j$'s are 0 and the constraint qualification says that $g^j(x^0) > 0$, $j = 1, \ldots, m$, it must be the case that at $x^0$ $\sum \lambda_j g^j(x^0) > 0$.

Contradicting the preceding. Hence, $\lambda_0 > 0$. We can therefore divide (14A-3) by $\lambda_0$, and if we define $\lambda^*_j = \frac{\lambda_j}{\lambda_0}$, $j = 1, \ldots, m$.

Eq. (14A-3) becomes

$$f(x) + \sum_{j=1}^{m} \lambda^*_j g^j(x) \leq f(x^*)$$

(14A-4)

When $x = x^*$, Eq. (14A-4) yields $\sum \lambda^*_j g^j(x^*) \leq 0$.

But since $\lambda^*_j \geq 0, g^j(x^*) \geq 0, j = 1, \ldots, m$,

$$\sum \lambda^*_j g^j(x^*) = 0$$

Defining the Lagrangian,

$$\mathcal{L}(x, \lambda) = f(x) + \sum_{j=1}^{m} \lambda_j g^j(x)$$

we find, with $x \geq 0, \lambda \geq 0$,

$$\mathcal{L}(x^*, \lambda^*) = f(x^*)$$

and therefore

$$\mathcal{L}(x, \lambda^*) = f(x) + \sum_{j=1}^{m} \lambda^*_j g^j(x) \leq f(x^*) = \mathcal{L}(x^*, \lambda^*)$$

(14A-5)

Satisfying the saddle point criterion. We showed in the chapter proper that

$$\mathcal{L}(x^*, \lambda^*) \leq \mathcal{L}(x, \lambda)$$

Hence, if $f(x), g^1(x), \ldots, g^m(x)$ are all concave and if there exists an $x^0$ such that $g^j(x^0) > 0, j = 1, \ldots, m$, solving the constrained maximum problem implies that the saddle point condition will be satisfied.

**BIBLIOGRAPHY**

The following articles and books all require advanced mathematical training.


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