Note. This problem may also be treated by noting that both sides of the equation satisfy the Helmholtz equation. The equality can be established by showing that the solutions have the same behavior at the origin and also behave alike at large distances. A "by inspection" type solution is developed in Section 16.6 using Green's functions.

12.4.8 Verify the Rayleigh equation of Exercise 12.4.7 by starting with the following steps:

1. Differentiate with respect to (kr) to establish

$$\sum_n a_n j_n'(kr) P_n(\cos\gamma) = i \sum_n a_n j_n(kr) \cos\gamma P_n(\cos\gamma).$$

2. Use a recurrence relation to replace $\cos\gamma P_n(\cos\gamma)$ by a linear combination of P_{n-1} and P_{n+1}.
3. Use a recurrence relation to replace j_n' by a linear combination of j_{n-1} and j_{n+1}.

12.4.9 From Exercise 12.4.7 show that

$$j_n(kr) = \frac{1}{2i^n} \int_{-1}^{1} e^{ikr\mu} P_n(\mu)\, d\mu.$$

This means that (apart from constant factors) the spherical Bessel function $j_n(kr)$ is the Fourier transform of the Legendre polynomial $P_n(\mu)$.

12.4.10 The Legendre polynomials and the spherical Bessel functions are related by

$$j_n(z) = \tfrac{1}{2}(-i)^n \int_0^{\pi} e^{iz\cos\theta} P_n(\cos\theta) \sin\theta\, d\theta, \qquad n = 0, 1, 2, \ldots.$$

Verify this relation by transforming the right-hand side into

$$\frac{z^n}{2^{n+1} n!} \int_0^{\pi} \cos(z\cos\theta) \sin^{2n+1}\theta\, d\theta$$

and using Exercise 11.7.9.

12.4.11 By direct evaluation of the Schlaefli integral show that $P_n(1) = 1$.

12.4.12 Explain why the contour of the Schlaefli integral, Eq. 12.69, is chosen to enclose the points $t = z$ *and* $t = 1$ when $n \to v$, not an integer.

12.4.13 In numerical work (such as the Gauss–Legendre quadrature of Appendix 2) it is useful to establish that $P_n(x)$ has n real zeros in the interior of $[-1, 1]$. Show that this is so.
Hint. Rolle's theorem shows that the first derivative of $(x^2 - 1)^{2n}$ has one zero in the interior of $[-1, 1]$. Extend this argument to the second, third, and ultimately to the nth derivative.

12.5 ASSOCIATED LEGENDRE FUNCTIONS

When Helmholtz's equation is separated in spherical polar coordinates (Section 2.6), one of the separated ordinary differential equations is the associated Legendre equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dP_n^m \cos\theta}{d\theta}\right) + \left[n(n+1) - \frac{m^2}{\sin^2\theta}\right] P_n^m(\cos\theta) = 0 \qquad (12.71)$$

With $x = \cos\theta$, this becomes

$$(1 - x^2)\frac{d^2}{dx^2}P_n^m(x) - 2x\frac{d}{dx}P_n^m(x) + \left[n(n+1) - \frac{m^2}{1 - x^2}\right]P_n^m(x) = 0 \quad (12.72)$$

Only if the azimuthal separation constant $m^2 = 0$ do we have Legendre's equation, Eq. 12.28.

One way of developing the solution of the associated Legendre equation is to start with the regular Legendre equation and convert it into the associated Legendre equation by using multiple differentiation. We take Legendre's equation

$$(1 - x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0, \quad (12.73)$$

and with the help of Leibnitz's formula[1] differentiate m times. The result is

$$(1 - x^2)u'' - 2x(m+1)u' + (n - m)(n + m + 1)u = 0, \quad (12.74)$$

where

$$u \equiv \frac{d^m}{dx^m}P_n(x). \quad (12.75)$$

Equation 12.74 is not self-adjoint. To put it into self-adjoint form, we replace $u(x)$ by

$$v(x) = (1 - x^2)^{m/2}u(x) = (1 - x^2)^{m/2}\frac{d^m P_n(x)}{dx^m}. \quad (12.76)$$

Solving for u and differentiating, we obtain

$$u' = \left(v' + \frac{mxv}{1 - x^2}\right)(1 - x^2)^{-m/2}, \quad (12.77)$$

$$u'' = \left[v'' + \frac{2mxv'}{1 - x^2} + \frac{mv}{1 - x^2} + \frac{m(m+2)x^2 v}{(1 - x^2)^2}\right]\cdot(1 - x^2)^{-m/2} \quad (12.78)$$

Substituting into Eq. 12.74, we find that the new function v satisfies the differential equation

$$(1 - x^2)v'' - 2xv' + \left[n(n+1) - \frac{m^2}{1 - x^2}\right]v = 0, \quad (12.79)$$

which is the associated Legendre equation reducing to Legendre's equation, as it must when m is set equal to zero. Expressed in spherical polar coordinates, the associated Legendre equation is

---

[1] Leibnitz's formula for the nth derivative of a product is

$$\frac{d^n}{dx^n}[A(x)B(x)] = \sum_{s=0}^{n}\binom{n}{s}\frac{d^{n-s}}{dx^{n-s}}A(x)\frac{d^s}{dx^s}B(x), \qquad \binom{n}{s} = \frac{n!}{(n-s)!\,s!},$$

a binomial coefficient.

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dv}{d\theta}\right)+\left[n(n+1)-\frac{m^2}{\sin^2\theta}\right]v=0. \tag{12.80}$$

Associated Legendre Functions

The regular solutions, relabeled $P_n^m(x)$, are

$$v \equiv P_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x). \tag{12.81}$$

These are the associated Legendre functions.[2] Since the highest power of x in $P_n(x)$ is x^n, we must have $m \le n$ (or the m-fold differentiation will drive our function to zero). In quantum mechanics the requirement that $m \le n$ has the physical interpretation that the expectation value of the square of the z-component of the angular momentum is less than or equal to the expectation value of the square of the angular momentum vector $\mathbf{L}$, $\langle L_z^2\rangle \le \langle L^2\rangle$.

From the form of Eq. 12.81 we might expect m to be nonnegative, differentiating a negative number of times not having been defined. However, if $P_n(x)$ is expressed by Rodrigues' formula, this limitation on m is relaxed and we may have $-n \le m \le n$, negative as well as positive values of m being permitted. Using Leibnitz's differentiation formula once again, the reader may show (Exercise 12.5.1) that

$P_n^m(x)$ and $P_n^{-m}(x)$ are related by

$$P_n^{-m}(x) = (-1)^m\frac{(n-m)!}{(n+m)!}P_n^m(x). \tag{12.81a}$$

From our definition of the associated Legendre functions, $P_n^m(x)$,

$$P_n^0(x) = P_n(x). \tag{12.82}$$

In addition, we may develop Table 12.3.

As with the Legendre polynomials, a generating function for the associated Legendre functions does exist:

$$\frac{(2m)!(1-x^2)^{m/2}}{2^m m!(1-2tx+t^2)^{m+1/2}} = \sum_{s=0}^{\infty} P_{s+m}^m(x)t^s. \tag{12.83}$$

However, because of its more cumbersome form and lack of any direct physical application, it is seldom used.

Recurrence Relations

As expected, the associated Legendre functions satisfy recurrence relations. Because of the existence of two indices instead of just one, we have a wide variety of recurrence relations:

---

[2] Occasionally (as in AMS-55), the reader will find the associated Legendre functions defined with an additional factor of $(-1)^m$. This $(-1)^m$ seems an unnecessary complication at this point. It will be included in the definition of the spherical harmonics $Y_n^m(\theta,\varphi)$ in Section 12.6.

TABLE 12.3 Associated Legendre Functions

$P_1^1(x) = (1 - x^2)^{1/2} = \sin\theta$
$P_2^1(x) = 3x(1 - x^2)^{1/2} = 3\cos\theta\sin\theta$
$P_2^2(x) = 3(1 - x^2) = 3\sin^2\theta$
$P_3^1(x) = \frac{3}{2}(5x^2 - 1)(1 - x^2)^{1/2} = \frac{3}{2}(5\cos^2\theta - 1)\sin\theta$
$P_3^2(x) = 15x(1 - x^2) = 15\cos\theta\sin^2\theta$
$P_3^3(x) = 15(1 - x^2)^{3/2} = 15\sin^3\theta$
$P_4^1(x) = \frac{5}{2}(7x^3 - 3x)(1 - x^2)^{1/2} = \frac{5}{2}(7\cos^3\theta - 3\cos\theta)\sin\theta$
$P_4^2(x) = \frac{15}{2}(7x^2 - 1)(1 - x^2) = \frac{15}{2}(7\cos^2\theta - 1)\sin^2\theta$
$P_4^3(x) = 105x(1 - x^2)^{3/2} = 105\cos\theta\sin^3\theta$
$P_4^4(x) = 105(1 - x^2)^2 = 105\sin^4\theta$

$$P_n^{m+1} - \frac{2mx}{(1 - x^2)^{1/2}} P_n^m + [n(n+1) - m(m-1)] P_n^{m-1} = 0, \tag{12.84}$$

$$(2n+1)xP_n^m = (n+m)P_{n-1}^m + (n-m+1)P_{n+1}^m, \tag{12.85}$$

$$\begin{aligned}(2n+1)(1 - x^2)^{1/2} P_n^m &\\ &= P_{n+1}^{m+1} - P_{n-1}^{m+1} \\ &= (n+m)(n+m-1)P_{n-1}^{m-1} - (n-m+1)(n-m+2)P_{n+1}^{m-1},\end{aligned} \tag{12.86}$$

$$(1 - x^2)^{1/2} P_n^{m\prime} = \tfrac{1}{2}P_n^{m+1} - \tfrac{1}{2}(n+m)(n-m+1)P_n^{m-1}. \tag{12.87}$$

These relations, and many other similar ones, may be verified by use of the generating function (Eq. 12.4), by substitution of the series solution of the associated Legendre equation (12.79) or reduction to the Legendre polynomial recurrence relations, using Eq. 12.81. As an example of the last method, consider the third equation in the preceding set. It is similar to Eq. 12.23:

$$(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x). \tag{12.88}$$

Let us differentiate this Legendre polynomial recurrence relation m times to obtain

$$\begin{aligned}(2n+1)\frac{d^m}{dx^m}P_n(x) &= \frac{d^m}{dx^m}P'_{n+1}(x) - \frac{d^m}{dx^m}P'_{n-1}(x) \\ &= \frac{d^{m+1}}{dx^{m+1}}P_{n+1}(x) - \frac{d^{m+1}}{dx^{m+1}}P_{n-1}(x).\end{aligned} \tag{12.89}$$

Now multiplying by $(1 - x^2)^{(m+1)/2}$ and using the definition of $P_n^m(x)$, we obtain Eq. 12.86.

Parity

The parity relation satisfied by the associated Legendre functions may be determined by examination of the defining equation (12.81). As $x \to -x$, we already know that $P_n(x)$ contributes a $(-1)^n$. The m-fold differentiation yields a factor of $(-1)^m$. Hence we have

$$P_n^m(-x) = (-1)^{n+m} P_n^m(x). \tag{12.90}$$

A glance at Table 12.3 verifies this for $1 \leq m \leq n \leq 4$.

Also, from the definition in Eq. 12.81

$$P_n^m(\pm 1) = 0, \qquad \text{for} \quad m \neq 0. \tag{12.91}$$

Orthogonality

The orthogonality of the $P_n^m(x)$ follows from the differential equation just as in $P_n(x)$ (Section 12.3); the term $-m^2/(1-x^2)$ cancels out, assuming m is the same in both cases. However, it is instructive to demonstrate the orthogonality by another method, a method that will also provide the normalization constant.

Using the definition in Eq. 12.81 and Rodrigues' formula (Eq. 12.65) for $P_n(x)$, we find

$$\int_{-1}^{1} P_p^m(x) P_q^m(x)\, dx = \frac{(-1)^m}{2^{p+q} p!\, q!} \int_{-1}^{1} X^m \frac{d^{p+m}}{dx^{p+m}} X^p \frac{d^{q+m}}{dx^{q+m}} X^q \, dx. \tag{12.92}$$

The function X is given by $X \equiv (x^2 - 1)$. If $p \neq q$, let us assume that $p < q$. Notice that the superscript m is the same for both functions. This is an essential condition. The technique is to integrate repeatedly by parts; all the integrated parts will vanish as long as there is a factor $X = x^2 - 1$. Let us integrate $q + m$ times to obtain

$$\int_{-1}^{1} P_p^m(x) P_q^m(x)\, dx = \frac{(-1)^m (-1)^{q+m}}{2^{p+q} p!\, q!} \int_{-1}^{1} \frac{d^{q+m}}{dx^{q+m}} \left(X^m \frac{d^{p+m}}{dx^{p+m}} X^p \right) X^q \, dx. \tag{12.93}$$

The integrand on the right-hand side is now expanded by Leibnitz's formula to give

$$X^q \frac{d^{q+m}}{dx^{q+m}} \left(X^m \frac{d^{p+m}}{dx^{p+m}} X^p \right) = X^q \sum_{i=0}^{i=q+m} \frac{(q+m)!}{i!\,(q+m-i)!} \frac{d^{q+m-i}}{dx^{q+m-i}} X^m \frac{d^{p+m+i}}{dx^{p+m+i}} X^p. \tag{12.94}$$

Since the term X^m contains no power of x greater than x^{2m}, we must have

$$q + m - i \leq 2m \tag{12.95}$$

or the derivative will vanish. Similarly,

$$p + m + i \leq 2p. \tag{12.96}$$

In the solution of these equations for the index i the conditions for a nonzero result are

$$i \geq q - m, \qquad i \leq p - m. \tag{12.97}$$

If $p < q$, as assumed, there is no solution and the integral vanishes. The same result obviously must follow if $p > q$.

For the remaining case, $p = q$, we may still have the single term corresponding to $i = q - m$. Putting Eq. 12.94 into Eq. 12.93, we have

$$\int_{-1}^{1} [P_q^m(x)]^2\,dx = \frac{(-1)^{q+2m}(q+m)!}{2^{2q}q!q!(2m)!(q-m)!}\int_{-1}^{1} X^q\left(\frac{d^{2m}}{dx^{2m}}X^m\right)\left(\frac{d^{2q}}{dx^{2q}}X^q\right)dx. \tag{12.98}$$

Since

$$X^m = (x^2-1)^m = x^{2m} - mx^{2m-2} + \cdots, \tag{12.99}$$

$$\frac{d^{2m}}{dx^{2m}}X^m = (2m)!, \tag{12.100}$$

Eq. 12.98 reduces to

$$\int_{-1}^{1} [P_q^m(x)]^2\,dx = \frac{(-1)^{q+2m}(2q)!(q+m)!}{2^{2q}q!q!(q-m)!}\int_{-1}^{1} X^q\,dx. \tag{12.101}$$

The integral on the right is just

$$(-1)^q\int_0^{\pi} \sin^{2q+1}\theta\,d\theta = \frac{(-1)^q 2^{2q+1}q!q!}{(2q+1)!} \tag{12.102}$$

(compare Exercise 10.4.9). Combining Eqs. 12.101 and 12.102, we have the orthogonality integral

$$\int_{-1}^{1} P_p^m(x)P_q^m(x)\,dx = \frac{2}{2q+1}\cdot\frac{(q+m)!}{(q-m)!}\delta_{p,q} \tag{12.103}$$

or, in spherical polar coordinates,

$$\int_0^{\pi} P_p^m(\cos\theta)P_q^m(\cos\theta)\sin\theta\,d\theta = \frac{2}{2q+1}\cdot\frac{(q+m)!}{(q-m)!}\delta_{p,q}. \tag{12.104}$$

The orthogonality of the Legendre polynomials is actually a special case of this result, obtained by setting m equal to zero; that is, for $m = 0$, Eq. 12.103 reduces to Eqs. 12.43 and 12.48. In both Eqs. 12.103 and 12.104 our Sturm–Liouville theory of Chapter 9 could provide the Kronecker delta. A special calculation, such as the analysis here, is required for the normalization constant.

The orthogonality of the associated Legendre functions over the same interval and with the same weighting factor as the Legendre polynomials does not contradict the uniqueness of the Gram–Schmidt construction of the Legendre polynomials, Example 9.3.1. Table 12.3 suggests (and Section 12.4 verifies) that $\int_{-1}^{1} P_p^m(x)P_q^m(x)\,dx$ may be written as

$$\int_{-1}^{1} p_p^m(x)p_q^m(x)(1-x^2)^m\,dx.$$

Here

$$p_p^m(x)(1-x^2)^{m/2} = P_p^m(x).$$

The functions $p_p^m(x)$ may be constructed by the Gram–Schmidt procedure with the weighting function $w(x) = (1 - x^2)^m$.

It is possible to develop an orthogonality relation for associated Legendre functions of the same lower index but different upper index. We find

$$\int_{-1}^{1} P_n^m(x) P_n^k(x)(1 - x^2)^{-1}\, dx = \frac{(n+m)!}{m(n-m)!}\delta_{m,k}. \qquad (12.105)$$

Note that a new weighting factor, $(1 - x^2)^{-1}$, has been introduced. This form is essentially a mathematical curiosity. In physical problems orthogonality of the φ dependence ties the two upper indices together and leads to Eq. 12.104.

EXAMPLE 12.5.1 Magnetic Induction Field of a Current Loop

Like the other differential equations of mathematical physics, the associated Legendre equation is likely to pop up quite unexpectedly. As an illustration, consider the magnetic induction field **B** and magnetic vector potential **A** created by a single circular current loop in the equatorial plane (Fig. 12.13).

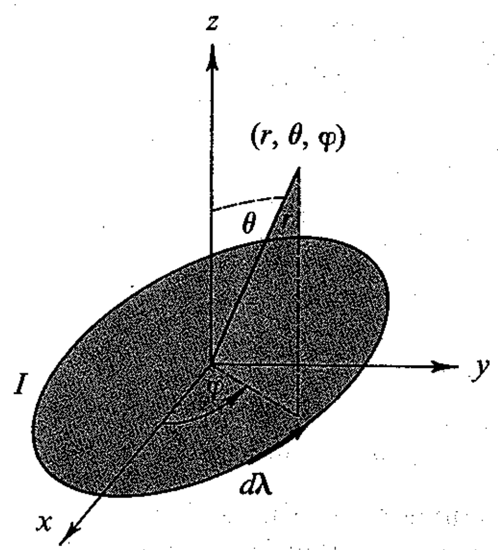

FIG. 12.13 Circular current loop

We know from electromagnetic theory that the contribution of current element $I\,d\lambda$ to the magnetic vector potential is

$$d\mathbf{A} = \frac{\mu_0}{4\pi}\frac{I\,d\boldsymbol{\lambda}}{r}. \qquad (12.106)$$

(This follows from Exercise 1.14.4). Equation 12.106, plus the symmetry of our system, shows that **A** has only a $\boldsymbol{\varphi}_0$-component and that the component is independent of φ[3]

$$\mathbf{A} = \boldsymbol{\varphi}_0 A_\varphi(r, \theta). \qquad (12.107)$$

[3] Pair off corresponding current elements $I\,d\lambda(\varphi_1)$ and $I\,d\lambda(\varphi_2)$, where $\varphi - \varphi_1 = \varphi_2 - \varphi$.

By Maxwell's equations

$$\nabla \times \mathbf{H} = \mathbf{J}, \qquad (\partial\mathbf{D}/\partial t = 0, \text{ SI units}). \tag{12.108}$$

Since

$$\mu_0 \mathbf{H} = \mathbf{B} = \nabla \times \mathbf{A}, \tag{12.109}$$

we have

$$\nabla \times \nabla \times \mathbf{A} = \mu_0 \mathbf{J}, \tag{12.110}$$

where **J** is the current density. In our problem **J** is zero everywhere except in the current loop. Therefore, away from the loop,

$$\nabla \times \nabla \times \boldsymbol{\varphi}_0 A_\varphi(r, \theta) = 0, \tag{12.111}$$

using Eq. 12.107.

From the expression for the curl in spherical polar coordinates (Section 2.5), we obtain (Example 2.5.2)

$$\nabla \times \nabla \times \boldsymbol{\varphi}_0 A_\varphi(r, \theta) = \boldsymbol{\varphi}_0 \left[-\frac{\partial^2 A_\varphi}{\partial r^2} - \frac{2}{r}\frac{\partial A_\varphi}{\partial r} - \frac{1}{r^2}\frac{\partial^2 A_\varphi}{\partial \theta^2} - \frac{1}{r^2}\frac{\partial}{\partial \theta}(\cot\theta A_\varphi) \right] = 0. \tag{12.112}$$

Letting $A_\varphi(r, \theta) = R(r)\Theta(\theta)$ and separating variables, we have

$$r^2 \frac{d^2 R}{dr^2} + 2r\frac{dR}{dr} - n(n+1)R = 0, \tag{12.113}$$

$$\frac{d^2\Theta}{d\theta^2} + \cot\theta \frac{d\Theta}{d\theta} + n(n+1)\Theta - \frac{\Theta}{\sin^2\theta} = 0. \tag{12.114}$$

The second equation is the associated Legendre equation (12.80) with $m = 1$, and we may immediately write

$$\Theta(\theta) = P_n^1(\cos\theta). \tag{12.115}$$

The separation constant $n(n+1)$ was chosen to keep this solution well behaved.

By trial, letting $R(r) = r^\alpha$, we find that $\alpha = n, -n-1$. The first possibility is discarded, for our solution must vanish as $r \to \infty$. Hence

$$A_{\varphi n} = \frac{b_n}{r^{n+1}} P_n^1(\cos\theta) = c_n \left(\frac{a}{r}\right)^{n+1} P_n^1(\cos\theta) \tag{12.116}$$

and

$$A_\varphi(r, \theta) = \sum_{n=1}^{\infty} c_n \left(\frac{a}{r}\right)^{n+1} P_n^1(\cos\theta), \qquad (r > a). \tag{12.117}$$

Here a is the radius of the current loop.

Since A_φ must be invariant to reflection in the equatorial plane by the symmetry of our problem,

$$A_\varphi(r, \cos\theta) = A_\varphi(r, -\cos\theta), \tag{12.118}$$

the parity property of $P_n^m(\cos\theta)$ (Eq. 12.90) shows that $c_n = 0$ for n even.

To complete the evaluation of the constants, we may use Eq. 12.117 to calculate B_z along the z-axis $[B_z = B_r(r, \theta = 0)]$ and compare with the expression obtained from the Biot and Savart law. This is the same technique that is used in Example 12.3.3. We have (compare Eq. 2.47)

$$\begin{aligned} B_r &= \nabla \times \mathbf{A}\big|_r \\ &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, A_\varphi)\right] \\ &= \frac{\cot\theta}{r}A_\varphi + \frac{1}{r}\frac{\partial A_\varphi}{\partial\theta}. \end{aligned} \tag{12.119}$$

Using

$$\begin{aligned} \frac{\partial P_n^1(\cos\theta)}{\partial\theta} &= -\sin\theta\frac{dP_n^1(\cos\theta)}{d(\cos\theta)} \\ &= -\frac{1}{2}P_n^2 + \frac{n(n+1)}{2}P_n^0 \end{aligned} \tag{12.120}$$

(Eq. 12.87) and then Eq. 12.84 with $m = 1$:

$$P_n^2(\cos\theta) - \frac{2\cos\theta}{\sin\theta}P_n^1(\cos\theta) + n(n+1)P_n(\cos\theta) = 0, \tag{12.121}$$

we obtain

$$B_r(r, \theta) = \sum_{n=1}^{\infty} c_n n(n+1)\frac{a^{n+1}}{r^{n+2}}P_n(\cos\theta), \qquad r > a \tag{12.122}$$

(for all θ). In particular, for $\theta = 0$,

$$B_r(r, 0) = \sum_{n=1}^{\infty} c_n n(n+1)\frac{a^{n+1}}{r^{n+2}}. \tag{12.123}$$

We may also obtain

$$\begin{aligned} B_\theta(r, \theta) &= -\frac{1}{r}\frac{\partial(rA_\varphi)}{\partial r} \\ &= \sum_{n=1}^{\infty} c_n n\frac{a^{n+1}}{r^{n+2}}P_n^1(\cos\theta), \qquad r > a. \end{aligned} \tag{12.124}$$

The Biot and Savart law states that

$$d\mathbf{B} = \frac{\mu_0}{4\pi}I\frac{d\boldsymbol{\lambda} \times \mathbf{r}_0}{r^2} \qquad \text{(SI units)}. \tag{12.125}$$

We now integrate over the perimeter of our loop (radius a). The geometry is shown in Fig. 12.14. The resulting magnetic induction field is $\mathbf{k}B_z$, along the z-axis, with

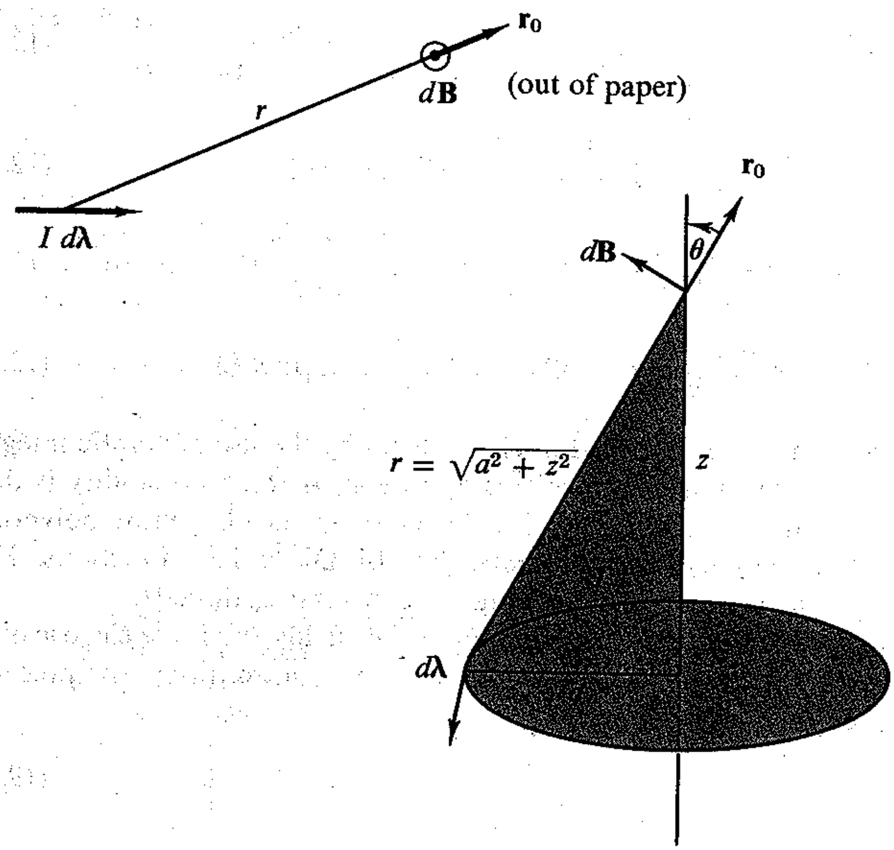

FIG. 12.14 Law of Biot and Savart applied to a circular loop

$$
\begin{aligned}
B_z &= \frac{\mu_0 I}{2} a^2 (a^2 + z^2)^{-3/2} \\
&= \frac{\mu_0 I}{2} \frac{a^2}{z^3} \left(1 + \frac{a^2}{z^2}\right)^{-3/2}.
\end{aligned}
\tag{12.126}
$$

Expanding by the binomial theorem, we obtain

$$
\begin{aligned}
B_z &= \frac{\mu_0 I}{2} \frac{a^2}{z^3} \left[1 - \frac{3}{2}\left(\frac{a}{z}\right)^2 + \frac{15}{8}\left(\frac{a}{z}\right)^4 - \cdots\right] \\
&= \frac{\mu_0 I}{2} \frac{a^2}{z^3} \sum_{s=0}^{\infty} (-1)^s \frac{(2s+1)!!}{(2s)!!} \left(\frac{a}{z}\right)^{2s}, \qquad z > a.
\end{aligned}
\tag{12.127}
$$

Equating Eqs. 12.123 and 12.127 term by term (with $r = z$),[4] we find

$$
\begin{aligned}
c_1 &= \frac{\mu_0 I}{4}, \qquad c_3 = -\frac{\mu_0 I}{16}, \qquad c_2 = c_4 = \cdots = 0. \\
c_n &= (-1)^{(n-1)/2} \frac{\mu_0 I}{2n(n+1)} \times \frac{(n/2)!}{[(n-1)/2]!(\frac{1}{2})!}, \qquad n \text{ odd}.
\end{aligned}
\tag{12.128}
$$

Equivalently, we may write

[4] The descending power series is also unique.

$$c_{2n+1} = (-1)^n \frac{\mu_0 I}{2^{2n+2}} \cdot \frac{(2n)!}{n!(n+1)!} = (-1)^n \frac{\mu_0 I}{2} \cdot \frac{(2n-1)!!}{(2n+2)!!} \tag{12.129}$$

and

$$A_\varphi(r, \theta) = \left(\frac{a}{r}\right)^2 \sum_{n=0}^{\infty} c_{2n+1} \left(\frac{a}{r}\right)^{2n} P_{2n+1}^1(\cos\theta), \tag{12.130}$$

$$B_r(r, \theta) = \frac{a^2}{r^3} \sum_{n=0}^{\infty} c_{2n+1}(2n+1)(2n+2)\left(\frac{a}{r}\right)^{2n} P_{2n+1}(\cos\theta), \tag{12.131}$$

$$B_\theta(r, \theta) = \frac{a^2}{r^3} \sum_{n=0}^{\infty} c_{2n+1}(2n+1)\left(\frac{a}{r}\right)^{2n} P_{2n+1}^1(\cos\theta). \tag{12.132}$$

These fields may be described in closed form by the use of elliptic integrals. Exercise 5.8.4 is an illustration of this approach. A third possibility is direct integration of Eq. 12.106 by expanding the factor $1/r$ as a Legendre polynomial generating function. The current is specified by Dirac delta functions. These methods have the advantage of yielding the constants c_n directly.

A comparison of magnetic current loop dipole fields and finite electric dipole fields may be of interest. For the magnetic current loop dipole the preceding analysis gives

$$B_r(r, \theta) = \frac{\mu_0 I}{2} \frac{a^2}{r^3}\left[P_1 - \frac{3}{2}\left(\frac{a}{r}\right)^2 P_3 + \cdots\right], \tag{12.133}$$

$$B_\theta(r, \theta) = \frac{\mu_0 I}{4} \frac{a^2}{r^3}\left[P_1^1 - \frac{3}{4}\left(\frac{a}{r}\right)^2 P_3^1 + \cdots\right]. \tag{12.134}$$

From the finite electric dipole potential of Section 12.1 we have

$$E_r(r, \theta) = \frac{qa}{\pi\varepsilon_0 r^3}\left[P_1 + 2\left(\frac{a}{r}\right)^2 P_3 + \cdots\right], \tag{12.135}$$

$$E_\theta(r, \theta) = \frac{qa}{2\pi\varepsilon_0 r^3}\left[P_1^1 + \left(\frac{a}{r}\right)^2 P_3^1 + \cdots\right]. \tag{12.136}$$

The two fields agree in form as far as the leading term is concerned ($r^{-3}P_1$), and this is the basis for calling them both dipole fields.

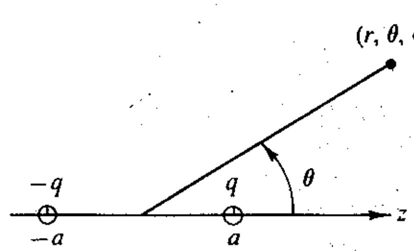

FIG. 12.15 Electric dipole

As with electric multipoles, it is sometimes convenient to discuss *point* magnetic multipoles. For the dipole case, Eqs. 12.133 and 12.134, the point dipole is formed by taking the limit $a \to 0$, $I \to \infty$ with Ia^2 held constant. With $\mathbf{n}$ a unit vector normal to the current loop (positive sense by right-hand rule, Section 1.10) the magnetic moment $\mathbf{m}$ is given by $\mathbf{m} = \mathbf{n}I\pi a^2$.